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## I.—ON A LAW EXISTING IN THE SUCCESSIVE APPROXIMATIONS TO A CONTINUED FRACTION.

LET there be a set of symbols $1_1$, $2_1$, $3_1$, $4_1$, &c. from which $1_2$, $2_2$, $3_2$, &c, $1_3$, $2_3$, $3_3$, &c., are formed as follows, $1_2 = 2_1\ 1_1 + 1$, $1_3 = 3_1\ 1_2 + 1_1$, $1_4 = 4_1\ 1_3 + 1_2$, &c., $2_2 = 3_1\ 2_1 + 1$, $2_3 = 4_1\ 2_2 + 2_1$, &c., so that the numerators and denominators of the successive approximations to the continued fraction $\frac{1}{n_1+}\ \frac{1}{n_2+}\ \frac{1}{n_3+}\ \ldots$ may be readily obtained in the usual manner.

The law of formation is

$$y_x = (y + x - 1)_1\ y_{x-1} + y_{x-2}.$$

Another law of formation is

$$y_x = y_n\ (y + n)_{x-n} + y_{n-1}\ (y + n + 1)_{x-n-1},$$

which may be easily proved. Though this double numerical symbol is convenient in the exhibition of laws, it will be better in ordinary work to write $a$, $b$, $c$, $d$, &c. for 1, 2, 3, 4, and to drop the suffixes in $a$, $b$, $c$, &c.

The object of this paper is the investigation of the law of formation exhibited in $a$, $ab + 1$, $abc + a + c$, &c., or $1_1\ 1_2\ 1_3$ &c. in terms of $1_1$, $2_1$, $3_1$, &c. Of these it is evident that the $n^{\text{th}}$ is made up of terms of $n, n-2, n-4, \ldots$ dimensions, the term of no dimension being always unity. If we signify by $(abc\ldots k)_n$ the collection of terms of the $n^{\text{th}}$ dimension which appear in the first step which introduces the letter $k$, we have for the several steps

$(a)_1$, $(ab)_2 + (ab)_0$, $(abc)_3 + (abc)_1$, $(abcd)_4 + (abcd)_2 + (abcd)_0$, and so on.

To find $(abc\ldots k)_n$ the rule is as follows. Take all those combinations of $a, b, \ldots k$, in which there are no breaks ex-

cept in pairs: thus in $(abcde)_3$ *acd* cannot occur, for a single letter *b* has dropt out by itself: but *cde, ade, abe, abc* can and do occur, and no others. Thus the terms of the third dimension in $1_5$ are $cde + ade + abe + abc$.

First, there can be no breaks but in pairs. When $k$ is introduced for the first time, the terms then factored with $k$ are factored with $l$ at the next step, and no break at all is made from $k$. But at the next step, the terms factored with $k$ two steps before, enter by simple addition, and are not factored with $m$. Consequently a term into which $k$ entered at the $n^{\text{th}}$ step, is never seen followed by $m$ without $l$, but has nothing before $n$ if $l$ be missing.

Secondly, all terms in which all breaks are pairs, or multiples of pairs, actually do enter. For instance, *ehim* must ultimately enter. It is clear from the formation that even steps must all contain unity, therefore the fifth step must contain *e*, which enters therefore by addition in the seventh step. Hence the eighth step has *eh* and the ninth *ehi*, which enters by addition into the eleventh step, and so the twelfth step has *ehim*, which is accordingly found in the fourteenth, sixteenth, &c. steps.

From these converse propositions it follows that, if $k$ represent the $m^{\text{th}}$ letter, the $m^{\text{th}}$ step, or ${}^1m$, is

$$(ab.\ldots.k)_m + (ab.\ldots.k)_{m-2} + (ab.\ldots.k)_{m-4} + \ldots\ldots$$

Thus the sixth step, written down independently of the preceding ones, is

$$abcdef + cdef + adef + abef + abcf + abcd + cf + cd + ad + ab + af + ef + 1.$$

When there are $m$ letters in the step, the number of terms of the $(m - 2n)^{\text{th}}$ dimension is the number of ways in which $m-2n$ can be taken out of $m-n$. And generally, the number of ways in which $n$ sets of $z$ consecutive letters each can be abstracted from $m$ letters, is the number of ways in which $m - zn$ (or $n$) can be removed out of $m - (z - 1)\,n$. To prove this, let $Z_{n,\,m}$ represent the number of ways just mentioned; in every one of these ways, either the last letter is among those abstracted, or it is not; $Z_{n-1,\,m-z}$ is the number of ways in which it is abstracted, $Z_{n,\,m-1}$ that in which it is not. Consequently

$$Z_{n,\,m} = Z_{n,\,m-1} + Z_{n-1,\,m-z}.$$

Let $m = nz + k$, whence $\Delta Z_{n-1,\,(n-1)\,z+k} = Z_{n,\,nz+k-1}$

$$\text{or} \quad Z_{n,\,nz+k} = \Sigma Z_{n+1,\,(n+1)\,z+k-1} \ldots\ldots$$

Now $Z_{n,\,nz} = 1$, and $Z_{1,\,z+k} = k + 1$; whence the successive operations by which to ascend from $Z_{n,nz}$ to $Z_{n,nz+k}$ for any value of $k$, are 1. Change $n$ into $n + 1$. 2. Perform the

operation of finite summation. 3. Determine the arbitrary constant so that $Z_{1,\,z+k}$ may be $k+1$. This gives us for values, beginning from $k=0$,

$$1,\ n+1,\ \frac{(n+1)(n+2)}{2},\ \frac{(n+1)(n+2)(n+3)}{2.3},\ \&c.,$$

which contain the theorem to be proved.

It appears then that in the $m^{\text{th}}$ approximation, the fourth dimension $(abc\ldots k)_m$ has one term only, but that $(abc\ldots k)_{m-2n}$ has

$$\frac{(m-n)(m-n-1).\ldots.(m-2n+1)}{1.2.3\ldots.n} \text{ terms.}$$

Accordingly, if $a=b=c=d$, &c., we have, for the $n^{\text{th}}$ step,

$$a^m+(m-1)a^{m-2}+\frac{m-2}{1}\frac{m-3}{2}a^{m-4}+\frac{m-3}{1}\frac{m-4}{2}\frac{m-5}{3}a^{m-6}+\ldots.$$

It may be worth the noting, that the sum of the coefficients just written down, or the value of the expression when $a=1$, is the number of distinct ways (different orders counting as different ways) in which $m+1$ can be compounded by addition of *odd* numbers; so that, $m$ increasing without limit, the number of ways in which $m$ can be made of odd numbers is to the same for $m+1$ ultimately in the ratio of the segments of a line which is divided in what Euclid calls extreme and mean ratio.

A. DE M.

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## II.—NOTES ON CONIC SECTIONS.

(1) THE eccentric anomaly may be made use of to prove several properties of the ellipse in a very simple manner, especially in all cases when conjugate diameters are concerned, as the following examples will shew.

There is a very simple but important theorem respecting the eccentric anomalies of the extremities of two conjugate diameters, which has not been noticed so far as I am aware. It is this:

*If $\phi$ and $\phi'$ be the eccentric anomalies of $P$ and $D$ respectively,* $\phi'-\phi=\frac{\pi}{2}$.

We may put the equation to the ellipse in the form of two equations, viz. $x=a\cos\phi,\quad y=b\sin\phi,$

the angle $\phi$ is called the eccentric anomaly of the point $xy$.

Now $xy$ and $x'y'$ being the co-ordinates of $P$ and $D$, and $\phi$, $\phi'$ the corresponding eccentric anomalies, we have

$$\left.\begin{aligned} x &= a\cos\phi, \quad y = b\sin\phi \\ x' &= a\cos\phi', \quad y' = b\sin\phi' \end{aligned}\right\} \ldots\ldots\ldots (1).$$

And, by a well known theorem,

$$\frac{y}{x}\frac{y'}{x'} = -\frac{b^2}{a^2},$$

$$\text{or} \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} = 0;$$

substituting the values (1) in this equation, we find immediately

$$\cos(\phi' - \phi) = 0,$$

and therefore $$\phi' - \phi = \frac{\pi}{2}.$$ Q. E. D.

(2) Let $r\theta$, $r'\theta'$, be the polar co-ordinates of $P$ and $D$, then

$$r^2 = x^2 + y^2 = a^2\cos^2\phi + b^2\sin^2\phi,$$

but by the theorem just proved $r'$ is obtained from $r$ by putting $\phi + \frac{\pi}{2}$ instead of $\phi$;

$$\therefore\ r'^2 = a^2\sin^2\phi + b^2\cos^2\phi,$$

$$\therefore\ r^2 + r'^2 = a^2 + b^2.$$

Again, $$\sin(\theta' - \theta) = \sin\theta'\cos\theta - \cos\theta'\sin\theta$$

$$= \frac{y'}{r'}\frac{x}{r} - \frac{x'}{r'}\frac{y}{r};$$

$$\therefore\ rr'\sin(\theta' - \theta) = y'x - x'y$$

$$= ab(\sin\phi'\cos\phi - \cos\phi'\sin\phi)$$

$$= ab.$$

since $$\phi' - \phi = \frac{\pi}{2},$$

which are two well known properties of conjugate diameters.

(3) If in the expressions $x = a\cos\phi$, $y = b\sin\phi$, we put $\phi + \frac{\pi}{2}$ for $\phi'$, we change $x$ and $y$ into $x'$ and $y'$, therefore

$$x' = -a\sin\phi = -\frac{a}{b}y,$$

$$y' = b\cos\phi = \frac{b}{a}x.$$

(4) To find the evolute to the ellipse, the equation to the normal being

$$y - y_1 = -\frac{dx_1}{dy_1}(x - x_1),$$

may be put in the form

$$y - b \sin \phi = \frac{a \sin \phi}{b \cos \phi} (x - a \cos \phi),$$

$$\text{or} \quad y = x \cdot \frac{a}{b} \tan \phi - \frac{a^2 - b^2}{b} \sin \phi.$$

Therefore, differentiating with respect to $\phi$, we have

$$0 = x \frac{a}{b} \frac{1}{\cos^2 \phi} - \frac{a^2 - b^2}{b} \cos \phi ;$$

$$\therefore \cos^3 \phi = \frac{x}{a} \quad \left( a = \frac{a^2 - b^2}{b} \right),$$

and changing $\phi$, $a$, $x$, into $\frac{\pi}{2} - \phi$, $b$, $y$,

$$\sin^3 \phi = \frac{y}{\beta} \quad \left( \beta = \frac{b^2 - a^2}{a} \right),$$

$$\therefore \left( \frac{x}{a} \right)^{\frac{2}{3}} + \left( \frac{y}{\beta} \right)^{\frac{2}{3}} = 1.$$

(5) By substituting $x = a \cos \phi$, $y = b \sin \phi$, in the equation of areas in central forces, viz.

$$(x - ae) \frac{dy}{dt} - y \frac{dx}{dt} = h,$$

we find immediately

$$ab \{(\cos \phi - e) \cos \phi + \sin^2 \phi\} \, d\phi = h dt,$$

$$\text{or} \quad (1 - e \cos \phi) \, d\phi = n dt \left( n = \frac{h}{ab} \right),$$

$$\therefore \phi - e \sin \phi = nt + \text{const.}$$

These few simple examples shew the use of the eccentric anomaly.

(6) To determine the magnitude and position of the axes of the conic section

$$Ax^2 + 2Bxy + Cy^2 = 1 \quad \ldots\ldots\ldots \quad (1),$$

the co-ordinates being oblique.

Let $\omega$ be the angle of ordination, then the equation to a circle having the origin as centre is

$$x^2 + 2xy \cos \omega + y^2 = r^2 \quad \ldots\ldots\ldots \quad (2).$$

(1) $r^2$ − (2) gives

$$(Ar^2 - 1) + 2 (Br^2 - \cos \omega) \frac{y}{x} + (Cr^2 - 1) \frac{y^2}{x^2} = 0 \ldots . (3).$$

If this equation gives two equal values of $\frac{y}{x}$, the circle (2) must touch the conic section (1), which can only be at the extremities of the axes. Therefore the condition that (3) may have equal roots, namely,

$$(Ar^2 - 1)(Cr^2 - 1) = (Br^2 - \cos\omega)^2 \ldots\ldots (4),$$

makes $r^2$ equal to either $a^2$ or $b^2$.

Moreover, if $z$ be the value of $\frac{y}{x}$ got from (3), the equation to either of the axes is

$$y = zx \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Hence $a^2$ and $b^2$ are the roots of

$$(AC - B^2)\, r^4 - (A + C + 2B\cos\omega)\, r^2 + \sin^2\omega = 0 \ldots (6);$$

and to determine the positions of the axes, since (3) is a perfect square, we have

$$(Cr^2 - 1)\frac{y}{x} + Br^2 - \cos\omega = 0,$$

$$(Ar^2 - 1)\frac{x}{y} + Br^2 - \cos\omega = 0.$$

Eliminating $r^2$ from these two equations, we find

$$(Cy - Bx)(x - y\cos\omega) - (Ax - By)(y - x\cos\omega) = 0,$$

which, by (5), is evidently the equation to the two axes considered as one locus.

(7) If $B = 0$ in (6), we have

$$r^4 - \left(\frac{1}{A} + \frac{1}{C}\right) r^2 + \frac{\sin^2\omega}{AC} = 0;$$

therefore, if we put $\frac{1}{A} = a'^2$, $\frac{1}{C} = b'^2$, we find

$$a^2 + b^2 = a'^2 + b'^2,$$

$$ab = a'b' \sin\omega.$$

M. O. B.

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## III.—ON THE THEORY OF ALGEBRAIC CURVES.

By A. Cayley, B.A. Fellow of Trinity College.

Suppose a curve defined by the equation $U = 0$, $U$ being a rational and integral function of the $m^{\text{th}}$ order of the co-ordinates $x, y$. It may always be assumed, without loss of generality, that the terms involving $x^m$, $y^m$, both of them appear in $(U)$; and also that the coefficient of $y^m$ is equal to unity. For in any particular curve where this was not the case, by transforming the axes, and dividing the new equation by the

coefficient of $y^m$, the conditions in question would become satisfied. Let $H_m$ denote the terms of $U$, which are of the order $(m)$, and let $y - ax$, $y - \beta x \ldots y - \lambda x$ be the factors of $H_m$. If the quantities $a, \beta \ldots \lambda$ are all of them different, the curve is said to have a number of asymptotic directions equal to the degree of its equation. Such curves only will be considered in the present paper, the consideration of the far more complicated theory of those curves, the number of whose asymptotic directions is less than the degree of their equation, being entirely rejected. Assuming, then, that the factors of $H_m$ are all of them different, we may deduce from the equation $U = 0$, by known methods, the series

$$y = ax + a' + \frac{a''}{x} + \ldots . \qquad (1).$$

$$y = \beta x + \beta' + \frac{\beta''}{x} + \ldots .$$

$$\vdots$$

$$y = \lambda x + \lambda' + \frac{\lambda''}{x} + \ldots .$$

And these being obtained, we have, identically,

$$U = (y - ax - a' - \ldots)(y - \beta x - \beta' - \ldots)\ldots(y - \lambda x - \lambda' - \ldots)\ldots (2).$$

The negative powers of $x$ on the second side, in point of fact, destroying each other. Supposing in general that $fx$ containing positive and negative powers of $x$, $Efx$ denotes the function which is obtained by the rejection of the negative powers, we may write

$$U = E\left(y - ax \ldots - \frac{a^{(m)}}{x^{m-1}}\right)\left(y - \beta x \ldots - \frac{\beta^{(m)}}{x^{m-1}}\right)\ldots\left(y - \lambda x \ldots - \frac{\lambda^{(m)}}{x^{m-1}}\right)\ldots (3),$$

the symbol $E$ being necessary in the present case, because, when the series are continued only to the power $x^{-m+1}$, the negative powers no longer destroy each other.

We may henceforward consider $U$ as originally given by the equation (3), the $m.(m+1)$ quantities $a, a' \ldots a^{(m)}, \beta, \beta' \ldots \beta^{(m)}, \ldots \lambda, \lambda' \ldots \lambda^{(m)}$ satisfying the equations obtained from the supposition that it is possible to determine the following terms $a^{(m+1)}, \beta^{(m+1)} \ldots \lambda^{(m+1)}, \ldots$ so that the terms containing negative powers of $x$, on the second side of equation (2), vanish. It is easily seen that $a, \beta \ldots \lambda, a', \beta' \ldots \lambda'$ are entirely arbitrary, $a'', \beta'' \ldots \lambda''$ satisfy a single equation involving only the preceding quantities, $a''', \beta''' \ldots \lambda'''$ two equations involving the quantities which precede them, and so on, until $a^{(m)}, \beta^{(m)} \ldots \lambda^{(m)}$, which satisfy $(m-1)$ relations, involving the preceding quan-

tities. Thus the $m.(m+1)$ quantities in question satisfy $\frac{m.m-1}{2}$ equations, or they may be considered as functions of $m.(m+1)-\frac{1}{2}m.(m-1)=\frac{1}{2}m.(m+3)$ arbitrary constants. Hence the value of $U$, given by the equation (3), is the most general expression for a function of the $m^{\text{th}}$ order. It is to be remarked also that the quantities $a^{(m+1)}, \beta^{(m+1)}..\lambda^{(m+1)},....$ are all of them completely determinable as functions of $a, \beta..\lambda, ..a^{(m)}, \beta^{(m)}..\lambda^{(m)}$.

The advantage of the above mode of expressing the function $U$, is the facility obtained by means of it for the elimination of the variable $(y)$ from the equation $U=0$, and any other analogous one $V=0$. In fact, suppose $V$ expressed in the same manner as $U$, or by the equation

$$V=E\left(y-Ax...-\frac{A^{(n)}}{x^{n-1}}\right)\left(y-Bx...-\frac{B^{(n)}}{x^{n-1}}\right)...\left(y-Kx...-\frac{K^{(n)}}{x^{n-1}}\right)...(4),$$

$n$ being the degree of the function $V$. It is almost unnecessary to remark, that $A, B..K,..A^{(n)}, B^{(n)}..K^{(n)}$ are to be considered as functions of $\frac{1}{2}n(n+3)$ arbitrary constants, and that the subsequent $A^{(n+1)}, B^{(n+1)}..K^{(n+1)}..$can be completely determined as functions of these. Determining the values of $y$ from the equation (3), viz. the values given by the equations (2); substituting these successively in the equation

$$V=(y-Ax-..)(y-Bx-..)..(y-Kx-..)=0...(5),$$

analogous to (2), and taking the product of the quantities so obtained, also observing that this product must be independent of negative powers of $x$, the result of the elimination may be written down under the form

$$E\left[\left\{(a-A)x..+\frac{a^{(mn)}-A^{(mn)}}{x^{mn-1}}\right\}...\left\{(a-Kx)...+\frac{a^{(mn)}-K^{(mn)}}{x^{mn-1}}\right\}\right]...(6),$$
$$\left[\times\left\{(\lambda-A)x..+\frac{\lambda^{(mn)}-A^{(mn)}}{x^{mn-1}}\right\}...\left\{(\lambda-Kx)...+\frac{\lambda^{(mn)}-K^{(mn)}}{x^{mn-1}}\right\}\right]$$

the series in $\{\ \}$ being continued only to $x^{-mn+1}$, because the terms after this point produce in the whole product nothing but terms involving negative powers of $x$. It is for the same reason that the series in $(\ )$, in the equations (3) and (4), are only continued to the terms involving $x^{-m+1}$, $x^{-n+1}$ respectively.

The first side of the equation (6) is of the order $mn$, in $x$, as it ought to be. But it is easy to see, from the form of the expression, in what case the order of the first side reduces

itself to a number less than $mn$. Thus, if $n$ be not greater than $m$, and the following equations be satisfied,

$$A = a,\ A^{(1)} = a^{(1)} \ldots A^{(r-1)} = a^{(r-1)}\ r \not> n. \ldots\ldots (7).$$
$$B = \beta,\ B^{(1)} = \beta^{(1)} \ldots B^{(s-1)} = \beta^{(s-1)}\ s \not> r$$
$$\vdots$$
$$K = \kappa,\ K^{(1)} = \kappa^{(1)} \ldots K^{(v-1)} = \kappa^{(v-1)}\ v \not> u.$$

The degree of the equation (6) is evidently $mn - r - s \ldots - v$, or the curves $U = 0$, $V = 0$ intersect in this number only of points. If $mn - r - s \ldots - v = 0$. The curves $U = 0$ and $V = 0$ do not intersect at all, and if $mn - r - s - v$ be negative, $= -\omega$ suppose, the equation (6) is satisfied identically, or the functions $U$, $V$ have a common factor, the number $\omega$ expressing the degree of this factor in $x$, $y$.

Supposing the function $V$ given arbitrary, it may be required to determine $U$, so that the curves $U = 0$, $V = 0$ intersect in a number $mn - k$ points. This may in general be done, and done in a variety of ways, for any value of $k$ from unity to $\frac{1}{2}\, m.(m + 3)$. I shall not discuss the question generally at present, nor examine into the meaning of the quantity $mn - \frac{1}{2}\, m \,.\, m + 3 \left\{= \frac{1}{2}\, m \,.\, (2n - m - 3)\right\}$ becoming negative, but confine myself to the simple case of $U$ and $V$, both of them functions of the second order. It is required, then, to find the equations of all those curves of the second order which intersect a given curve of the second order in a number of points less than four.

Assume, in general,

$$V = E\left(y - Ax - A' - \frac{A''}{x}\right)\left(y - Bx - B' - \frac{B''}{x}\right),$$

$A''$, $B''$ satisfy $A'' + B'' = 0$, and putting $B'' = \dfrac{K}{A - B}$, and $\therefore A'' = -\dfrac{K}{A - B}$, and reducing

$$V = (y - Ax - A')(y - Bx - B') + K.$$

Similarly $U = E\left(y - ax - a' - \dfrac{a''}{x}\right)\left(y - \beta x - \beta' - \dfrac{\beta''}{x}\right)$,

$a'' + \beta'' = 0$, and putting $\beta'' = \dfrac{k}{a - \beta}$, and $\therefore a'' = -\dfrac{k}{a - \beta}$, and reducing

$$U = (y - ax - a')(y - \beta x - \beta') + k.$$

Suppose

(1) $U = 0$, $V = 0$ intersect in three points, we must have $a = A$, or the curve $U = 0$ must have one of its asymptotes parallel to one of the asymptotes of $V = 0$.

(2) The curves intersect in two points. We must have $a = A$, $a' = A'$, or else $a = A$, $\beta = B$; i.e. $U = 0$ must have one of its asymptotes coincident with one of the asymptotes of the curve $V = 0$, or else it must have its asymptotes each of them parallel to $V = 0$. The latter case is that of similar and similarly situated curves.

(3) Suppose the curves intersect in a single point only. Then either $a = A$, $a' = A'$, $a'' = A''$, which it is easy to see gives

$$U = (y - Ax - A')(y - \beta x - \beta') + K \cdot \frac{A - \beta}{A - B},$$

or else $a = A$, $a' = A'$, $\beta = B$, which is the case of one of the asymptotes of the curve $U = 0$, coinciding with one of those of the curve $V = 0$, and the remaining asymptotes parallel. As for the first case, if $a$, $a_1$ are the transverse axes, $\theta$, $\theta_1$ the inclination of the two asymptotes to each other, these four quantities are connected by the equation

$$\frac{a^2}{a_1^2} = \frac{\tan\theta \cdot \cos^2\frac{\theta}{2}}{\tan\theta_1 \cos^2\frac{\theta_1}{2}};$$

and besides, one of the asymptotes of the first curve is coincident with one of the asymptotes of the second. This is a remarkable case; it may be as well to verify that $U = 0$, $V = 0$ do intersect in a single point only. Multiplying the first by $y - Bx - B'$, the second by $y - \beta x - \beta'$, and subtracting, the result is

$$(A - \beta)(y - Bx - B') - (A - B)(y - \beta x - \beta') = 0,$$

reducible to

$$y - Ax = \frac{A(B' - \beta') + B\beta' - B'\beta}{B - \beta}, \quad \text{i.e. } y - Ax - C = 0.$$

Combining this with $V = 0$, we have an equation of the form $y - Bx - D = 0$. And from this and $y - Ax - C$, $x$, $y$ are determined by means of a simple equation.

(4) Lastly, when the curves do not intersect at all. Here $a = A$, $a' = A'$, $\beta = B$, $\beta' = B'$, or the asymptotes of $U = 0$ coincide with those of $V = 0$; *i.e.* the curves are similar, similarly situated, and concentric; or else $a = A$, $a' = A'$, $a'' = A''$, $\beta = B$. Here

$$U = (y - Ax - A')(y - Bx - \beta') + K,$$

or the required curve has one of its asymptotes coincident with one of those of the proposed curve; the remaining two

asymptotes are parallel, and the magnitudes of the curves are equal.

In general, if two curves of the orders $m$ and $n$, respectively, are such that $r$ asymptotes of the first are parallel to as many of the second, $s$ out of these asymptotes being coincident in the two curves, the number of points of intersection is $mn - r - s$; but the converse of this theorem is not true.

In a former paper, *On the Intersection of Curves*, I investigated the number of arbitrary constants in the equation of a curve of a given order $(\rho)$ subjected to pass through the $mn$ points of intersection of two curves of the orders $m$ and $n$ respectively. The reasoning there employed is not applicable to the case where the two curves intersect in a number of points less than $mn$. In fact, it was assumed that, $W = 0$ being the equation of the required curve, $W$ was of the form $uU + vV$; $u$, $v$ being polynomials of the degrees $\rho - m$, $\rho - n$ respectively. This is, in point of fact, true in the case there considered, viz. that in which the two curves intersect in $mn$ points; but where the number of points of intersection is less than this, $u$, $v$ may be assumed polynomials of an order *higher* than $\rho - m$, $\rho - n$, and yet $uU + vV$ reduce itself to the order $(\rho)$. The preceding investigations enable us to resolve the question for every possible case.

Considering then the functions $U$, $V$ determined as before by the equations (3), (4). Suppose, in the first place, we have a system of equations

$$\alpha = A,\ \beta = B. \ldots \theta = H \ (t \text{ equations}) \ldots . \quad (8).$$

Assume $P = (y - \alpha x - \ldots)(y - \beta x - \ldots)\ldots(y - \theta x - \ldots)$,

$$Q = (y - Ax - \ldots)(y - Bx - \ldots)\ldots(y - Hx - \ldots).$$

$$\Upsilon = (y - \iota x - \ldots)\ldots(y - \lambda x - \ldots),$$

$$\Psi = (y - \mathrm{I}x \ldots)\ldots(y - Kx - \ldots);$$

whence $U = P\Upsilon, \quad V = Q\Psi.$

Suppose $\Upsilon = E\Upsilon + \Delta\Upsilon, \quad \Psi = E\Psi + \Delta\Psi,$

$$E\Psi.U - E\Upsilon.V = E\Psi.P\Upsilon - E\Upsilon.Q\Psi,$$

$$= E\Psi.P.(E\Upsilon + \Delta\Upsilon) - E\Upsilon.Q\,(E\Psi + \Delta\Psi),$$

$$= E\Upsilon.E\Psi.(P - Q) + E\Psi.P.\Delta\Upsilon - E\Upsilon.Q.\Delta\Psi,$$

$$= E\{E\Upsilon.E\Psi.(P - Q) + E\Psi.P\Delta\Upsilon - E\Upsilon.Q.\Delta\Psi\},$$

$$= \Pi \text{ suppose.}$$

In this expression $E\Upsilon$, $E\Psi$ are of the degrees $m - t$, $n - t$, $\Delta\Upsilon$, $\Delta\Psi$ of the degree $(-1)$, and $P$, $Q$, $P - Q$ of the degrees $t$, $t$, $(t - 1)$ respectively. The terms of $\Pi$ are therefore of the

degrees $m + n - t - 1$, $(m - 1)$, $(n - 1)$ respectively, and the largest of these is in general $m + n - t - 1$. Suppose, however, $m + n - t - 1$ should be equal to $(m - 1)$ (it cannot be inferior to it), then $t = n$. $\Psi$ becomes equal to unity, or $\Delta\Psi$ vanishes. The remaining two terms of $\Pi$ are $E\Upsilon\,(P - Q)$, $P\Delta\Upsilon$, which are of the degrees $(m - 1)$, $(n - 1)$ respectively, $\Pi$ is still of the degree $(m - 1)$, supposing $m > n$. If $m = n$, the term $P\Delta\Upsilon$ vanishes. $\Pi$ is still of the degree $m - 1$. Hence in every case the degree of $\Pi$ is $m + n - t - 1$. (Assuming always that $(P - Q)$ does not reduce itself to a degree lower than $t = 1$, which is always the case as long as the equations $\alpha' = A'$, $\beta' = B' \ldots . \theta' = H'$ are not all of them satisfied simultaneously). It will be seen presently that we shall gain in symmetry by wording the theorem thus: the degree of $\Pi$ is equal to the greatest of the two quantities $m + n - t - 1$, $m - 1$.

Suppose next, in addition to the equations (8), we have

$$\alpha' = A',\ \beta' = B' \ldots . \zeta' = F', \quad t' \text{ equations } t' \not> t \ldots (8').$$

Then, taking $\Upsilon'$, $\Psi'$, $P'$, $Q'$ the analogous quantities to $\Upsilon$, $\Psi$, $P$, $Q$, and putting $E\Psi' . U - E\Upsilon' . V = \Pi'$,

we have, as before,

$$\Pi' = E\{E\Upsilon' . E\Psi' . (P' - Q') + E\Psi' . P'\Delta\Upsilon' - E\Upsilon' . Q'\Delta\Psi' .\}.$$

The degree of $P' - Q'$ is $t' - 2$ (unless simultaneously $\alpha'' = A''$, $\beta'' = B'' \ldots . \zeta'' = F''$, in which case the degree may be lower). The degrees, therefore, of the terms of $\Pi'$ are $m + n - t' - 2$, $n - 1$, $m - 1$. Or we may say that the degree of $\Pi'$ is equal to the greatest of the quantities $m + n - t' - 2$, $m - 1$; though to establish this proposition in the case where $t' = (n - 1)$ would require some additional considerations.

Continuing in this manner until we come to the quantity $\Pi^{(k-1)}$, the degree of this quantity is the greatest of the two numbers $m + n - t^{(k-1)} - k$, $(m - 1)$. And we may suppose that none of the equations $\alpha^{(k)} = A^{(k)} \ldots .$ are satisfied, so that the series $\Pi, \Pi' \ldots . \Pi^{(k-1)}$ is not to be continued beyond this point.

Considering how the equation of the curve passing through the $mn - t - t' \ldots - t^{(k-1)}$ points of intersection of $U = 0$, $V = 0$. We may write

$$W = uU + vV + p\Pi + p'\Pi' \ldots . + p^{(k-1)}\,\Pi^{(k-1)} = 0 \ldots . (9)$$

for the required equation; the dimensions of $u, v, p, p' \ldots p^{(k-1)}$ being respectively

$$\rho - m,\ \rho - n,\ \rho - m - n + t + 1 \text{ or } \rho - m + 1,\ \rho - m - n + t' + 2$$
$$\text{or } \rho - m + 1, \ldots . \rho - m - n + t^{(k-1)} + k \text{ or } \rho - m + 1,$$

the lowest of the two numbers being taken for the dimensions of $p, p' \ldots p^{k-1}$. Also, if any of these numbers become negative, the corresponding term is to be rejected. In saying that the degrees of $p, p' \ldots p^{(k-1)}$ have these actual values, it is supposed that the degrees of $\Pi, \Pi' \ldots \Pi^{k-1}$ actually ascend to the greatest of the values

$\rho-m-n+t+1$, or $(m-1)$, $m+n-t'-2$, or $(m-1)$, $-m+n-t^{(k-1)}+k$, or $(m-1)$.

The cases of exception to this are when several of the consecutive numbers $t, t' \ldots t^{(k-1)}$ are equal. In this case the corresponding terms of the series $\Pi, \Pi' \ldots \Pi^{k-1}$, are also equal. Suppose for instance $t, t'$ were equal, $\Pi, \Pi'$ would also be equal. A term of $p$ of an *order higher* by unity than $\rho - m - n + t + 1$, or $\rho - m + 1$, which is the highest term admissible, produces in $p\Pi$ a term, identical with one of the terms of $p'\Pi$. So that nothing is gained in generality by admitting such terms into $p$. The equation (9), with the preceding values for the dimensions of $p, p' \ldots p^{(k-1)}$, may be employed, therefore, even when several consecutive terms of the sines $t, t' \ldots t^{(k-1)}$ are equal. It will be convenient also to assume that $\rho - m$ is not negative, or at least for greater simplicity to examine this case in the first place.

$u, U$, and $v, V$, contain terms of the form $x^\alpha y^\beta U$, $x^\gamma y^\delta V$, $\alpha + \beta \not> \rho - m$, $\gamma + \delta \not> \rho - n$. $p\Pi$ contains terms of this form, and $m$ addition terms where $\alpha + \beta = (\rho + 1 - m)\ \gamma + \delta = (\rho + 1 - n)$. It is useless to repeat the former terms, so we may assume for $p$, a *homogeneous* function of the order $\rho - m - n + t + 1$, or $\rho - m + 1$; in which case $p\Pi$ consists only of terms where $\alpha + \beta = (\rho + 1 - m)$, $\gamma + \delta = (\rho + 1 - n)$. And the general expression of $p$ contains $\rho - m - n + t + 2$, or $\rho - m + 2$, arbitrary constants. Similarly $p'\Pi'$ contains terms of the form of those in $uU$, $vV$, $p\Pi$, and also terms for which

$$\alpha + \beta = (\rho + 2 - m),\ \gamma + \delta = \rho + 2 - n.$$

The latter terms only need be considered, or $p'$ may be assumed to be a homogeneous function of the order $\rho - m - n + t' + 2$, or $\rho - m + 1$, containing therefore $\rho - m - n - t' + 3$, or $(\rho - m) + 2$ arbitrary constants.

Similarly $p^{(k-1)}$ contains $\rho - m - n + t^{(k-1)} + k + 1$ or $\rho - m + 2$ arbitrary constants. Assume

$$\nabla = \binom{\rho-m-n+t+2}{\rho-m+2} + \binom{\rho-m-n+t'+3}{\rho-m+2} \ldots + \binom{\rho-m-n+t^h+k+1}{\rho-m+2} \ldots (10),$$

where, in forming the value of $\nabla$ the least of the two quantities in ( ) is to be taken, this value also, if negative, being replaced by zero. The number of arbitrary constants in $p, p' \ldots p^{(k-1)}$ is consequently equal to $\Delta$.

The number of arbitrary constants in $u$, $v$, are respectively $\{1 + 2. \ldots .(\rho - m + 1)\}$ and $\{1 + 2. \ldots .(\rho - n + 1)\}$ i.e.

$$\tfrac{1}{2}(\rho - m + 1)(\rho - m + 2), \text{ and } \tfrac{1}{2}(\rho - n + 1)(\rho - n + 2).$$

Or the whole number of arbitrary constants in $W$, diminished by unity (since nothing is gained in generality, by leaving the coefficient, for instance of $y^\rho$ indeterminate, instead of supposing it equal to unity) becomes

$$\tfrac{1}{2}(\rho - m + 1)(\rho - m + 2) + \tfrac{1}{2}.(\rho - n + 1)(\rho - n + 2) + \nabla - 1,$$

reducible to

$$\tfrac{1}{2}\rho.(\rho + 3) + \tfrac{1}{2}.(\rho - m - n + 1)(\rho - m - n + 2) - mn + \nabla.$$

By the reasonings contained in the paper already referred to, if $\rho + k - m - n + 1$ be positive, to find the number of really disposable constants in $W$, we must subtract from this number a number $\frac{1}{2}(\rho + k - m - n + 1)(\rho + k - m - n + 2)$. Hence, calling $\phi$ the number of disposable constants in $W$, we have

$$\phi = \tfrac{1}{2}\rho.(\rho + 3) + \tfrac{1}{2}(\rho - m - n + 1)(\rho - m - n + 2) - mn + \nabla - \Lambda \ldots (11),$$

where $\Lambda = 0$, if $\rho + k - m - n + 1$ be negative or zero ... (12),

$$\Lambda = \tfrac{1}{2}(\rho + k - m - n + 1)(\rho + k - m - n + 2),$$

if $\rho + k - m - n + 1$ be positive.

And $\nabla$ is given by the equation (9). Also, if $\theta$ be the number of points through which the curve $W = 0$ can be drawn, *including* the points of intersection of the curves $U = 0$, $V = 0$, $\theta = \phi + (mn - t - t' \ldots . - t^{k-1})$ or

$$\theta = \tfrac{1}{2}\rho.(\rho + 3) + \tfrac{1}{2}(\rho - m - n + 1)(\rho - m - n + 2) + \nabla - \Lambda - t - t' \ldots - t^{k-1} \ldots (13).$$

Any particular cases may be deduced with the greatest facility from these general formulæ. Thus, supposing the curves to intersect in the complete number of points $mn$, we have

$$\phi = \tfrac{1}{2}\rho.(\rho + 3) + \tfrac{1}{2}(1 - \delta)(\rho - m - n + 1)(\rho - m - n + 2) - mn,$$

$\delta$ being zero or unity according as $\rho < (m + n - 1)$ or $\rho > (m + n - 1)$. Reducing, we have, for $\rho \not> m + n - 3$,

$$\phi = \tfrac{1}{2}\rho.(\rho + 3) + \tfrac{1}{2}(\rho - m - n + 1)(\rho - m - n + 2) - mn,$$
$$\theta = \tfrac{1}{2}\rho.(\rho + 3) + \tfrac{1}{2}(\rho - m - n + 1)(\rho - m - n + 2).$$

And for $\rho > m + n - 3$,

$$\phi = \tfrac{1}{2}\rho.(\rho + 3) - mn,$$
$$\theta = \tfrac{1}{2}\rho.(\rho + 3).$$

Suppose, in the next place, the curves have $t$ parallel pairs of asymptotes, none of these pairs being coincident. Then $\rho \not> m + n - t - 2$,

$$\phi = \tfrac{1}{2}\rho.(\rho + 3) + \tfrac{1}{2}(\rho - m - n + 1)(\rho - m - n + 2) - mn,$$
$$\theta = \tfrac{1}{2}\rho.(\rho + 3) + \tfrac{1}{2}(\rho - m - n + 1)(\rho - m - n + 2) - t.$$

$\rho > m + n - t - 2$, $\rho \not> m + n - 2$,

$\phi = \frac{1}{2}\rho.(\rho + 3) + \frac{1}{2}(\rho - m - n + 2)(\rho - m - n + 3) - mn + t$,

$\theta = \frac{1}{2}\rho.(\rho + 3) + \frac{1}{2}(\rho - m - n + 2)(\rho - m - n + 3)$,

$\rho > m + n - 2$, $\phi = \frac{1}{2}.\rho.(\rho + 3) - mn + t$,

$\theta = \frac{1}{2}\rho.(\rho + 3)$.

In which, if $t$ be equal to 2 or greater than 2, the limiting conditions are more conveniently written

$\rho \not> m + n - t - 2$; $\rho \not> m + n - t - 2 > m + n - 4$; $\rho > m + n - 4$.

Similarly may the solution of the question be explicitly obtained when the curves have $t$ asymptotes parallel, and $t'$ out of these coincident, but the number of separate formulæ will be greater.

In conclusion, I may add the following references to two memoirs on the present subject: the conclusions in one point of view are considerably less general even than those of my former paper, though much more so in another. Jacobi Theoremata de punctis intersect. duar. curvar. algeb.; Crelle's Journal, vol. xv.; Plucker Theoremes sur les equations a plusieurs variables, d° vol. xvi.

*Addition.*

As an exemplification of the preceding formulæ, and besides, as a question interesting in itself, it may be proposed to determine the asymptotic curves of the $r^{\text{th}}$ order of a given curve, having all its asymptotic directions distinct,—$r$ being any number less than the degree of the equation of the given curve.

Def$^{\text{n}}$. A curve of the $r^{\text{th}}$ order, which intersects a given curve of the $m^{\text{th}}$ order in a number of points, $= mr - \frac{r.(r+3)}{2}$, is said to be an asymptotic curve of the $r^{\text{th}}$ order to the curve in question. Suppose, as before, $U = 0$ being the equation to the given curve,

$$U = E\left(y - ax - \ldots - \frac{a^{(m)}}{x^{m-1}}\right)\ldots\left(y - \lambda x \ldots - \frac{\lambda^{(n)}}{x^{m-1}}\right);$$

and let $\theta, \phi \ldots \omega$ denote any combination of $r$ terms out of the series $a \ldots \lambda \ldots \theta', \phi' \ldots \omega'$, &c. . . the corresponding terms out of $a' \ldots \lambda'$, &c. Then, writing

$$V = E\left\{\left(y - \theta x \ldots - \frac{\theta^{(m)}}{x^{m-1}}\right)\left(y - \phi x \ldots - \frac{\phi^{(m-1)}}{x^{m-2}} - \frac{\Phi^{(m)}}{x^{m-1}}\right) \times \ldots .\right.$$

$$\left.\left(y - \psi x \ldots - \frac{\psi^{(m-2)}}{x^{m-3}} - \frac{\Psi^{(m-1)}}{x^{m-2}} - \frac{\Psi^{(m)}}{x^{m-1}}\right)\ldots\left(y - \omega x - \Omega' \ldots - \frac{\Omega^{(m)}}{x^{m-1}}\right)\right\},$$

(where the quantities $\Phi^{(m)}$, $\Psi^{(m-1)}$, $\Psi^{(m)}..\Omega'..\Omega^{(m)}$ are entirely determinate, since, by what has preceded, $\theta'$, $\phi'..\Omega'$ satisfy a certain equation $\theta'', \phi''..\Omega''$ two equations. $\ldots\theta^{(m)}\Phi^{(m)}..\Omega^{(m)}$ $(m-1)$ equations), we have

$$V = 0$$

for the required equation of the asymptotic curve. It is obvious that the whole number of asymptotic curves of the order $r$, is $n.(n-1)..(n-r+1)$, viz. $1.2..r$ curves for each combination of $\frac{n.(n-1)..(n-r+1)}{1.2..r}$ asymptotes. Some particular instances of asymptotic curves will be found in a memoir by M. Plucker, *Liouville's Journal*, vol. 1. on the Enumeration of Curves of the Fourth Order. The general theory does not seem to be one to which much attention has been paid.

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## IV.—THE POLAR EQUATION TO THE TANGENT TO A CONIC SECTION.

LET the polar equation to the conic section be

$$\frac{c}{r} = 1 + e\cos\theta.$$

Let the equation to the tangent be

$$\frac{c}{r} = m\cos\theta + n\sin\theta;$$

therefore at the point of contact the two values of $\sin\theta$, given by the equation

$$1 + e\cos\theta = m\cos\theta + n\sin\theta,$$

are each $= \sin\alpha$, if $\alpha$ be the value of $\theta$ at the point of contact; and because $(1 - n\sin\theta)^2 = (m-e)^2(1-\sin^2\theta)$, by the condition of equal roots,

$$\{n^2 + (m-e)^2\}\{1 - (m-e)^2\} = n^2,$$

$$\therefore (m-e)^2 + n^2 = 1,$$

$$\therefore \sin\alpha - n = 0,$$

$$\cos\alpha - (m-e) = 0;$$

and the equation to the tangent is

$$\frac{c}{r} = e\cos\theta + \cos(\theta - \alpha).$$

If (fig. 1) $PT$, $QT$ be tangents at points $P$ and $Q$ of a conic section whose equation is $\frac{c}{r} = 1 + e\cos\theta$; $\alpha$, $\beta$ the

spiral angles at $P$ and $Q$, reckoned from the pole, the equations to $PT$, $QT$ are

$$\frac{c}{r} = \cos(\theta - \alpha) + e\cos\theta,$$

$$\text{and } \frac{c}{r} = \cos(\theta - \beta) + e\cos\theta;$$

therefore at the point of intersection

$$\cos(\theta - \alpha) = \cos(\theta - \beta),$$

$$\theta - \alpha = \beta - \theta,$$

or $ST$ bisects the angle $PSQ$.

To find the locus of the intersection of a perpendicular to the focal distance from the focus with the tangent.

In this case the equation to the tangent is

$$\frac{c}{r} = \cos(\theta - \alpha) + e\cos\theta,$$

equation to the perpendicular supposed

$$\theta = \alpha - \frac{\pi}{2};$$

therefore, eliminating $\alpha$, the equation to the locus required is

$$\frac{c}{r} = e\cos\theta,$$

$$\text{or } r\cos\theta = \frac{c}{e},$$

the equation to the directrix.

To find the locus of the intersection of two tangents at extremities of focal distances which make equal angles with the latus rectum.

Here $$\alpha = \pi - \beta,$$

$$\theta = \frac{\alpha + \beta}{2} = \frac{\pi}{2},$$

or the locus is the latus rectum.

(Fig. 2) The enunciation of this problem is given in p. 171, vol. I. The axes are $AKL.ARQ$.

Let the equation to $HK$ be $a_1x + b_1y = 1$.....(1),

.....$HL$....$a_2x + b_2y = 1$.....(2),

.....$PQ$....$a_1x + \beta_1y = 1$.....(3),

.....$PR$....$a_2x + \beta_2y = 1$.....(4),

.....$KQ$....$a_1x + \beta_1y = 1$.....(5),

.....$RL$....$a_2x + \beta_2y = 1$.....(6),

where $a_1$ $b_1$ &c. are the reciprocals of the intercepts for the different lines. $H$, $O$, and $P$ are in a straight line.

The co-ordinates of $H$ are given by (1) and (2),

..... $P$ ..... (3) and (4),

..... $O$ ..... (5) and (6).

Multiply (1) and (5) by $h$,

(2) and (6) by $k$.

Then, by subtraction, we have

$$\text{at } H \quad (a_1h - a_2k)\,x + (b_1h - b_2k)\,y = h - k,$$

$$\text{at } O \quad (a_1h - a_2k)\,x + (\beta_1h - \beta_2k)\,y = h - k;$$

$$\text{and if } b_1h - b_2k = \beta_1h - \beta_2k,$$

$$\text{or } (b_1 - \beta_1)\,h = (b_2 - \beta_2)\,k,$$

the relations are the same, or either is the equation to the straight line joining $HO$;

$$\text{at } P \quad (a_1h - \beta_1k)\,x + (\beta_1h - \beta_2k)\,y = h - k,$$

$$\therefore \quad a_1h - a_2k = a_1h - a_2k,$$

$$\text{or} \quad (a_1 - a_1)\,h = (a_2 - a_2)\,k;$$

$$\therefore \quad \frac{a_1 - a_1}{b_1 - \beta_1} = \frac{a_2 - a_2}{b_2 - \beta_2} \quad \ldots\ldots\ldots\ldots (A).$$

Again, at $C$, $\quad (a_1 - a_1)\,x + (b_1 - \beta_1)\,y = 0,$

at $B$, $\quad (a_2 - a_2)\,x + (b_2 - \beta_2)\,y = 0;$

which two equations, by $(A)$, are equations to the same straight line which passes through the origin $A$.

If $ABCD$ (fig. 3) be any conic section, $AB$ any chord of the conic section, $ACBD$ a quadrilateral on $AB$, $P$ the intersection of $AD$, $BC$, $Q$ the intersection of the diagonals; $PQ$ always passes through the point $O$, which is the intersection of two tangents.

Let $$OA = \frac{1}{a}, \quad OB = \frac{1}{b};$$

equation to $AD$ $\quad ax + \beta y = 1 \ldots\ldots\ldots\ldots (1),$

..... $BC$ $\quad \gamma x + by = 1 \ldots\ldots\ldots\ldots (2),$

..... $BD$ $\quad ax + by = 1 \ldots\ldots\ldots\ldots (3),$

..... $AC$ $\quad ax + \delta y = 1 \ldots\ldots\ldots\ldots (4);$

equation to the conic section,

$$(ax + by - 1)^2 = Babxy,$$

$$\text{or} \quad ax + by - 1 = \sqrt{(Babxy)};$$

at $D$ the intersection of (1) and (3)

$$ax + \beta y - 1 = 0;$$

therefore $(b - \beta)\, y = \sqrt{(Babxy)}$,

or $(b - \beta)^2\, y = Babx.$

Also $(a - \alpha)\, x = (b - \beta)\, y,$

therefore $(a - \alpha)(b - \beta) = Bab.$

Similarly $(a - \gamma)(b - \delta) = Bab,$

Hence $$\frac{a - \alpha}{b - \delta} = \frac{a - p}{b - \beta} \quad \ldots\ldots\ldots\ldots (A).$$

Also at $P$ $(a - \gamma)\, x = (b - \beta)\, y,$

at $Q$ $(a - \alpha)\, x = (b - \delta)\, y;$

which two relations coincide by $(A)$, and either is the equation to $PQ$, which therefore passes through $O$.

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## V.—DEMONSTRATION OF A PROPOSITION IN PHYSICAL OPTICS.

To shew that the planes of polarization of the waves in a biaxal crystal bisect the angles between the planes through the normal to the front and the wave axes.

If $l, m, n$ be the direction cosines of a wave axis, the axes of elasticity being the axes of co-ordinates, it is shewn that

$$l = \pm \sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)}, \quad m = 0, \quad n = \sqrt{\left(\frac{b^2 - c^2}{a^2 - c^2}\right)}.$$

Let $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ be the direction cosines of the lines of rectilinear vibration, $\theta_1\theta_2$, $\phi_1\phi_2$ the angles which the wave axes make with each of these lines; then

$$\frac{\cos\theta_1}{\cos\theta_2} = \frac{\alpha_1\sqrt{(a^2 - b^2)} + \gamma_1\sqrt{(b^2 - c^2)}}{\alpha\sqrt{(a^2 - b^2)} - \gamma_1\sqrt{(b^2 - c^2)}} = \frac{A\frac{\alpha_1}{\gamma_1} + 1}{A\frac{\alpha_1}{\gamma_1} - 1},$$

where $$A = \sqrt{\left(\frac{a^2 - b^2}{b^2 - c^2}\right)}.$$

Similarly $$\frac{\cos\phi_1}{\cos\phi_2} = \frac{A\frac{\alpha_2}{\gamma_2} + 1}{A\frac{\alpha_2}{\gamma_2} - 1},$$

$$\therefore \frac{\cos\theta_1}{\cos\theta_2} \cdot \frac{\cos\phi_2}{\cos\phi_1} = \frac{A^2 \frac{a_1 a_2}{\gamma_1\gamma_2} + A\left(\frac{a_2}{\gamma_2} - \frac{a_1}{\gamma_1}\right) - 1}{A^2 \frac{a_1 a_2}{\gamma_1\gamma_2} - A\left(\frac{a_2}{\gamma_2} - \frac{a_1}{\gamma_1}\right) - 1}.$$

Now it is shewn (*Mathematical Journal*, vol. I. p. 79), that

$$\frac{a_1 a_2}{\gamma_1\gamma_2} = \frac{1}{A^2};$$

$$\text{hence} \quad \frac{\cos\theta_1}{\cos\theta_2} = -\frac{\cos\phi_1}{\cos\phi_2};$$

which shows that the projections of the wave axes on the plane of the wave make equal angles with the directions of rectilinear vibration. For let $\chi$, $\psi$ be the angles the wave axes make respectively with the normal to the front, $\omega$, $\omega'$ the angles their projections on the front make with one of the directions of rectilineal vibration, then it is easily seen that

$$\cos\theta_1 = \sin\chi\cos\omega, \quad \cos\phi_1 = \sin\psi\cos\omega',$$

$$\cos\theta_2 = \sin\chi\sin\omega, \quad \cos\phi_2 = \sin\psi\sin\omega',$$

and therefore the relation just proved becomes

$$\cot\omega = -\cot\omega', \quad \text{or} \quad \omega = -\omega',$$

since $\omega$ and $\omega'$ may be considered acute angles. And from hence the truth of the proposition is evident.

This proof appeared under a more complicated form in *Liouville's Mathematical Journal*, Sept. 1843.

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## VI.—ON A MULTIPLE DEFINITE INTEGRAL.

By R. L. Ellis, M.A. Fellow of Trinity College.

In the eighteenth number of the Journal, I pointed out the mode in which Fourier's theorem may be employed in the evaluation of certain definite multiple integrals. The theorem generally known as Liouville's, and another of the same degree of generality, were readily deduced from the considerations then suggested. I proceed to another application of the same method.

Theor. $$\int dx \int dy \ldots \frac{f(mx + ny + \ldots)}{(a^2 + x^2)(b^2 + y^2)\ldots}$$

$$= \pi^{\nu-1} \frac{ma + nb + \ldots}{ab \ldots} \int_h^{h'} \frac{fu\,du}{u^2 + (ma + nb + \ldots)^2};$$

the integral being supposed to involve $\nu$ variables $x$, $y$, &c., and the limits being given by the inequalities

$$mx + ny + \ldots \underset{>}{=} h \text{ and } \underset{=}{<} h'.$$

(Negative as well as positive values of the variables are admissible.)

Dem. Recurring to the general theorem stated at the commencement of the paper already mentioned, we see that the integral whose value is sought is equal to

$$\frac{1}{\pi}\int_h^{h'} fu\,du \int_0^{\infty} da \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \ldots \frac{\cos a\,(mx + ny + \ldots - u)}{(a^2 + x^2)(b^2 + y^2)\ldots}.$$

Develope the cosine in a series of products of sines and cosines of simple arcs. Every term involving a sine disappears on integration, as the limits extend from $-\infty$ to $+\infty$. Consequently the last written expression becomes

$$\frac{1}{\pi}\int_h^{h'} fu\,du \int_0^{\infty} \cos au\,da \int_{-\infty}^{+\infty} \frac{\cos amx}{a^2 + x^2}\,dx \int_{-\infty}^{+\infty} \frac{\cos any}{b^2 + y^2}\,dy \ldots.$$

Now $$\int_{-\infty}^{+\infty} \frac{\cos amx}{a^2 + x^2}\,du = \frac{\pi}{a}\,e^{-ma x} \text{ &c.} = \text{&c.},$$

and thus the integral becomes

$$\frac{\pi^{\nu-1}}{ab\ldots}\int_h^{h'} fudu \int_0^{\infty} \cos aue^{-(ma+nb+\ldots)x}\,da,$$

$$\text{or,}\quad \pi^{\nu-1}\,\frac{ma + nb + \ldots}{ab\ldots}\int_h^{h'} \frac{fu\,du}{u^2 + (ma + nb + \ldots)^2};$$

which was to be proved.

It may be well to verify this result in a particular case. Let $\nu = 2$; then we have to prove that

$$\int dx \int dy\,\frac{f(mx + ny)}{(a^2 + x^2)(b^2 + y^2)} = \pi\,\frac{ma + nb}{ab}\int_h^{h'} \frac{fu\,du}{u^2 + (ma + nb)^2};$$

for simplicity, we will suppose that

$$m^2 + n^2 = 1.$$

Let $mx + ny = u$, and therefore $x = mu + nv$,

$nx - my = v$, $y = nu - mv$,

$u$ and $v$ being two new variables. As $u^2 + v^2 = x^2 + y^2$, $dxdy$ is to be replaced by $dudv$, and thus

$$\int dx \int dy\,\frac{f(mx + ny)}{(a^2 + x^2)(b^2 + y^2)} \text{ becomes}$$

$$\int f du \int \frac{dv}{\{a^2 + (mu + nv)^2\}\{b^2 + (nu - mv)^2\}}.$$

The limits are easily seen to be $+\infty$ $-\infty$ for $v$; $h'$ and $h$ for $u$. For the integral expresses the volume of that portion of a solid bounded by the surface, whose equation is

$$z = \frac{f(mx + ny)}{(a^2 + x^2)(b^2 + y^2)};$$

which is included between the plane of $(xy)$, the bounding surface, and two planes parallel to one another and perpendicular to $(xy)$. The equations of these planes are respectively

$$mx + ny = h, \text{ and } mx + ny = h',$$

and the introduction of $u$ and $v$ is equivalent to changing the axes of co-ordinates, so that one of the new axes, that of $u$, is perpendicular to these planes, while the other is parallel to them.

In order to find the value of

$$\int_{-\infty}^{+\infty} \frac{dv}{\{a^2 + (mu + nv)^2\}\{b^2 + (nu - mv)^2\}}, \text{ assume}$$

$$\frac{1}{\{a^2 + (mu + nv)^2\}\{b^2 + (nu - mv)^2\}}$$

$$= \frac{A(mu + nv) + Bna}{a^2 + (mu + nv)^2} + \frac{C(nu - mv) + Dmb}{b^2 + (nu - mv)^2}.$$

It is evident that the terms in $A$ and $C$ will disappear on integration between infinite limits: those in $B$ and $D$ become respectively $\pi B$ and $\pi D$, and the integral in question is therefore

$$\pi(B + D).$$

Now it may be shown that

$$B = \frac{n}{a} \frac{u^2 - m^2a^2 + n^2b^2}{\{u^2 + (ma + nb)^2\}\{u^2 + (ma - nb)^2\}};$$

$$D = \frac{m}{b} \frac{u^2 + m^2a^2 - n^2b^2}{\{u^2 + (ma + nb)^2\}\{u^2 + (ma - nb)^2\}}.$$

Consequently $B + D = \frac{ma + nb}{ab} \frac{1}{u^2 + (ma + nb)^2}$;

and thus the integral sought is seen to be equal to

$$\pi \frac{ma + nb}{av} \int_h^{h'} \frac{f u du}{u^2 + (ma + nb)^2},$$

which was to be proved.

Similar considerations apply in the case of more variables, and doubtless by induction our general result might be established. But the method we have followed, besides being more analytical, is also very much simpler.

Another result of this same kind of analysis, I shall indicate without a demonstration, which there will be no difficulty in supplying.

$$\int dx \int dy \ldots e^{-a^2x^2-b^2y^2\ldots} f(max + nby + \ldots)$$

$$= \frac{\pi^{\frac{n-1}{2}}}{ab\ldots} \frac{1}{\{m^2 + n^2 + \ldots\}^{\frac{1}{2}}} \int_h^{h'} e^{-\frac{u^2}{m^2+n^2+\ldots}} f u du,$$

the limits being given by

$$max + nby + \ldots \underset{>}{=} h \text{ and } \underset{=}{<} h'.$$

In conclusion, it may be well to remark, that the analysis of which we have made use is not unfrequently applicable to questions which though not difficult in principle are nevertheless somewhat perplexing in practice.

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## VII.—CHAPTERS IN THE ANALYTICAL GEOMETRY OF (*n*) DIMENSIONS.

By A. Cayley, B.A. Fellow of Trinity College.

Chap. 1. *On some preliminary formulæ.*

I take for granted all the ordinary formulæ relating to determinants. It will be convenient, however, to write down a few, relating to a certain system of determinants, which are of considerable importance in that which follows: they are all of them either known, or immediately deducible from known formulæ.

Consider the series of terms

$$\begin{matrix} x_1, & x_2 \ldots x_n & \ldots\ldots\ldots\ldots\ldots (1), \\ A_1, & A_2 \ldots A_n & \\ \vdots & \vdots & \\ K_1, & K_2 \ldots K_n & \end{matrix}$$

the number of the quantities $A \ldots K_1$ being equal to $r$ $(r < n)$. Suppose $(r + 1)$ vertical rows selected, and the quantities contained in them formed into a determinant, this may be done in $\frac{n.(n-1)\ldots(q+2)}{1.2\ldots n-q-1}$ different ways. The

system of determinants so obtained will be represented by the notation

$$\left\| \begin{matrix} x_1, & x_2 \ldots . & x_n \\ A_1, & A_2 \ldots . & A_n \\ \vdots & & \vdots \\ K_1, & K_2 \ldots . & K_n \end{matrix} \right\} \quad \ldots\ldots\ldots\ldots (2).$$

And the system of equations, obtained by equating each of these determinants to zero, by the notation

$$\left\| \begin{matrix} x_1, & x_2 \ldots . & x_n \\ A_1, & A_2 \ldots . & A_n \\ \vdots & & \vdots \\ K_1, & K_2 \ldots . & K_n \end{matrix} \right\} = 0 \ldots\ldots\ldots (3).$$

The $\dfrac{n.(n-1)\ldots.(q+2)}{1.2\ldots.(n-q+1)}$ equations represented by this formula reduce themselves to $(n-q)$ independent equations. Imagine these expressed by

$$(1) = 0, \quad (2) = 0 \ldots\ldots (n-q) = 0 \ldots\ldots (4).$$

Any one of the determinants of (2) is reducible to the form

$$\Theta_1 (1) + \Theta_2 (2) \ldots + \Theta_{n-q} (n-q) \ldots\ldots\ldots (5),$$

where $\Theta_1, \Theta_2 \ldots \Theta_{n-9}$ are coefficients independent of $x_1, x_2 \ldots x_n$. The equations (3) may be replaced by

$$\left\| \begin{matrix} \lambda_1 x_1 + \lambda_2 x_2 + \ldots \lambda_n x_n, & \mu_1 x_1 + \ldots, & \ldots \tau_1 x_1 + \ldots \\ \lambda_1 A_1 + \lambda_2 A_2 + \ldots \lambda_n x_n, & \mu_1 A_1 + \ldots, & \ldots \tau_1 A_1 + \ldots \\ \vdots & & \\ \lambda_1 K_1 + \lambda_2 K_2 + \ldots \lambda_n K_n, & \mu_1 K_1 + \ldots, & \tau_1 K_1 + \ldots \end{matrix} \right\} = 0 \ldots (6).$$

And conversely from (6) we may deduce (3), unless

$$\left| \begin{matrix} \lambda_1, & \lambda_2 \ldots . & \lambda_n \\ \mu_1, & \mu_2 \ldots . & \mu_n \\ \vdots & & \\ \tau_1, & \tau_2 \ldots . & \tau_n \end{matrix} \right| = 0 \ldots\ldots\ldots\ldots (7).$$

(The number of the quantities $\lambda, \mu \ldots \tau$ is of course equal to $n$). The equations (3) may also be expressed in the form

$$\left\| \begin{matrix} & x_1, & x_2, & \ldots & x_n \\ \lambda_1 A_1 + \ldots \omega_1 K_1, & \lambda_1 A_2 + \ldots \omega_1 K_2, & \ldots \lambda_1 A_n \ldots + \omega_1 K_n \\ \vdots & \vdots & \\ \lambda_r A_1 + \ldots \omega_r K_1, & \lambda_r A_2 + \ldots \omega_r K_2, & \ldots \lambda_r A_n \ldots + \omega_r K_n \end{matrix} \right\} \ldots (8).$$

The number of the quantities $\lambda, \mu \ldots \omega$ being $(r)$.

And conversely (3) is deducible from (8), unless

$$\begin{vmatrix} \lambda_1 \ldots \omega_1 \\ \vdots \\ \lambda_r \ldots \omega_r \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

CHAP. 2. *On the determination of linear equations in $x_1, x_2 \ldots x_n$ which are satisfied by the values of these quantities derived from given systems of linear equations.*

It is required to find linear equations in $x_1 \ldots x_n$ which are satisfied by the values of these quantities derived—1. from the equations $\mathfrak{A}' = 0$, $\mathfrak{B}' = 0 \ldots \mathfrak{G}' = 0$; 2. from the equations $\mathfrak{A}'' = 0$, $\mathfrak{B}'' = 0 \ldots \mathfrak{N}'' = 0$; 3. from $\mathfrak{A}''' = 0$, $\mathfrak{B}''' = 0$ $\ldots \mathfrak{K}''' = 0$, &c. &c., where

$$\mathfrak{A} = A_1 x_1 + A_2 x_2 \ldots + A_n x_n \ldots\ldots\ldots (1).$$
$$\mathfrak{B} = B_1 x_1 + B_2 x_2 \ldots + B_n x_n .$$
$$\vdots$$

Also $r'$, $r''$... representing the number of equations in the systems (1), (2)... and $k$ the number of these given systems,

$$(n - r') + (n - r'') + \ldots \not> n - 1 \text{ or } (k - 1)\, n + 1 \not> r' + r'' + \ldots$$

Assume $$C\Psi = \lambda'\mathfrak{A}' + \mu'\mathfrak{B}' + \ldots\ldots\ldots\ldots\ldots (2).$$
$$\lambda'\mathfrak{A}' + \mu'\mathfrak{B}' + \ldots =$$
$$\lambda''\mathfrak{A}'' + \mu''\mathfrak{B}'' + \ldots =$$
$$\lambda'''\mathfrak{A}''' + \mu'''\mathfrak{B}''' + \ldots =$$
$$\&c.$$

The latter equations being satisfied for the terms involving $x_1$, for those involving $x_2$, &c...separately. Suppose, in addition to these, a set of linear equations in $\lambda', \mu' \ldots \lambda'' \mu'' \ldots$ so that, with the preceding ones, there is a sufficient number of equations for the elimination of these quantities. Then, performing the elimination, the value of $\mathfrak{N}$, so obtained, is a function of $x_1, x_2 \ldots$ which vanishes for the values of these quantities, derived from the equations (1) or (2)...&c. The series of equations $\Psi = 0$ may be expressed in the form

$$\left\| \begin{array}{cccccccc} \mathfrak{A}' & \mathfrak{B}' \ldots & \mathfrak{C}' & & & & & \\ A'_1 & B'_1 \ldots & G'_1 & A''_1 \ldots & O''_1 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ A'_n & B'_n \ldots & G'_n & A''_n \ldots & O''_n & & & \\ & & & A''_1 \ldots & O''_1 & A'''_1 \ldots & R'''_1 & \\ & & & \vdots & \vdots & \vdots & \vdots & \\ & & & A''_n \ldots & O''_n & A'''_n \ldots & R'''_n & \\ & & & & & \cdot & \cdot \quad \cdot \quad \cdot & \\ & & & & & \cdot & \cdot \quad \cdot \quad \cdot & \end{array} \right\| = 0 \ldots\ldots (3).$$

CHAP. 3. *On reciprocal equations.*

Consider a system of equations

$$A_1x_1 + A_2x_2 \ldots + A_nx_n = 0 \ldots\ldots\ldots (1),$$
$$\vdots$$
$$K_1x_1 + K_2x_2 \ldots + K_nx_n = 0$$

($r$ in number).

The reciprocal system with respect to a given function ($U$) of the second order in $x_1, x_2 \ldots x_n$, is said to be

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \ldots d_{x_n}U \\ A_1, & A_2 \ldots\ldots A_n \\ \vdots & \\ K_1, & K_2 \ldots\ldots K_n \end{matrix} \right\| = 0 \ldots\ldots\ldots (2),$$

($n - r$ in number).

It must first be shewn that the reciprocal system to (2) is the system (1), or that the systems (1), (2) are reciprocals of each other.

Consider, in general, the system of equations

$$a_1d_{x_1}U + a_2d_{x_2}U \ldots + a_nd_{x_n}U = 0 \ldots\ldots\ldots (3).$$
$$\vdots$$
$$\lambda_1d_{x_1}U + \lambda_2d_{x_2}U \ldots + \lambda_nd_{x_n}U = 0.$$

Suppose $2U = \Sigma(a^2)\, x_a{}^2 + 2\Sigma(a\beta)\, x_ax_b,$

so that $d_{x_s}U = \Sigma(sa)\, x_a \ldots\ldots\ldots\ldots (4), (5).$

The equations (3) may be written

$$x_1\{a_1(1^2) + a_2(12) \ldots + a_n(1n)\} + \ldots$$
$$\text{&c.} \qquad + x_n\{a_1(n1) + a_2.(n2) \ldots + a_n(n^2)\} = 0 \ldots (6).$$

And forming the reciprocals of these, also replacing $d_{x_1}U$, $d_{x_2}U \ldots$ by their values, we have

$$\left\| \begin{matrix} x_1(1^2) + x_2(12) + \ldots x_n(1n) \ldots x_1(n1) + x_2(n.2) \ldots + x_n(n^2) \\ a_1(1^2) + a_2(12) + \ldots a_n(1.n) \ldots a_1(n1) + a_2(n.2) \ldots + a_n(n^2) \\ \vdots \\ \lambda_1(1^2) + \lambda_2(12) + \ldots \lambda_n(1.n) \ldots \lambda_1(n1) + \lambda_2(n2) \ldots + \lambda_n(n^2) \end{matrix} \right\| = 0 \ldots (7).$$

From which, assuming

$$\left\| \begin{matrix} (1^2)\ (12) \ldots (1n) \\ (21).(2^2) \ldots (2n) \\ \vdots \\ (n1)\ (n2) \ldots (n^2) \end{matrix} \right\| = 0 \ldots\ldots\ldots (8).$$

We have, for the reciprocal system of (3),

$$\left\| \begin{matrix} x_1, & x_2 \ldots x_n \\ a_1, & a_2 \ldots a_n \\ \vdots & \\ \lambda_1, & \lambda_2 \ldots \lambda_n \end{matrix} \right\| = 0 \ldots\ldots\ldots\ldots (9).$$

Now, suppose the equations (3) represent the system (2); their number in this case must be $(n-r)$. Also if $\theta$ represent any one of the quantities $\alpha, \beta \ldots \lambda$, we have

$$\begin{matrix} A_1\theta_1 + A_2\theta_2 \ldots + A_n\theta_n = 0 \ldots\ldots\ldots (10). \\ \vdots \\ K_1\theta_1 + K_2\theta_2 \ldots + K_n\theta_n = 0. \end{matrix}$$

By means of these equations, the system (9) may be reduced to the form

$$\left\| \begin{matrix} A_1x_1+A_2x_2\ldots+A_nx_n\ldots K_1x_1+K_2x_2\ldots+K_nx_n . x_{r+1}, & x_{r+2}\ldots x_n \\ 0 \quad\ldots\quad 0 & a_{r+1}, a_{r+2}\ldots a_n \\ \vdots & \vdots \\ 0 \quad\ldots\quad 0 & \lambda_{r+1}, \lambda_{r+2}\ldots\lambda_n \end{matrix} \right\| = \ldots (11),$$

which are satisfied by the equations (1). Hence the reciprocal system to (2) is (1), or (1), (2) are reciprocals to each other.

THEOREM. Consider the equations

$$\begin{matrix} (\mathfrak{A}' = 0, \quad \mathfrak{B}' = 0 \ldots \mathfrak{G}' = 0) \ldots\ldots\ldots (12), \\ (\mathfrak{A}'' = 0, \quad \mathfrak{B}'' = 0 \ldots \mathfrak{N}'' = 0), \\ (\mathfrak{A}''' = 0, \quad \mathfrak{B}''' = 0 \ldots \mathfrak{K}''' = 0), \\ \text{\&c.} \end{matrix}$$

of Chap. 2. The equations

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \ldots d_{x_n}U \\ A'_1, & A'_2 \ldots A'_n \\ \vdots & \\ G'_1, & G'_2 \ldots G'_n \end{matrix} \right\| = 0 \ldots\ldots\ldots (13),$$

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \ldots d_{x_n}U \\ A''_1, & A''_2 \ldots A''_n \\ \vdots & \\ O''_1, & O''_2 \ldots O''_n \end{matrix} \right\| = 0,$$

&c.

which are the reciprocals of these systems, represent taken conjointly, the reciprocal of the system of equations (3) of the same chapter.

Let this system, which contains $n - \{(n-r) + (n-r') + \ldots\}$ equations, be represented by

$$\begin{aligned} &a_1x_1 + a_2x_2 \ldots + a_nx_n = 0 \ldots\ldots\ldots (14). \\ &\beta_1x_1 + \beta_2x_2 \ldots + \beta_nx_n = 0. \\ &\vdots \\ &\zeta_1x_1 + \zeta_2x_2 \ldots + \zeta_nx_n = 0. \end{aligned}$$

The reciprocal system is

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \ldots d_{xn}U \\ a_1, & a_2 \ldots a_n \\ \vdots & \\ \zeta_1, & \zeta_2 \ldots \zeta_n \end{matrix} \right\| = 0 \ldots\ldots\ldots (15).$$

containing $(n-r) + (n-r') +$ &c. ... equations.

Also, by the formulæ in Chap. 2,

$$\begin{aligned} &a_1x_1 + \ldots + a_nx_n = \lambda'_1\mathfrak{A}' + \mu'_1\mathfrak{B}' + \ldots\sigma'_1\mathfrak{G}' \quad (\lambda, \mu \ldots \sigma, r' \text{ in number}). \\ &\beta_1x_1 + \ldots + \beta_nx_n = \lambda'_2\mathfrak{A}' + \mu'_2\mathfrak{B}' + \ldots\sigma'_2\mathfrak{G}' \\ &\vdots \\ &\zeta_1x_1 \quad \ldots + \zeta_nx_n = \lambda'_\theta\mathfrak{A}' + \mu'_\theta\mathfrak{B}' + \ldots\sigma'_\theta\mathfrak{G}' \ldots\ldots\ldots\ldots\ldots (16), \end{aligned}$$

writing $\theta = n - \{(n-r) + (n-r') + \ldots\}$.

Also, assuming any arbitrary quantities $\eta_1, \eta_2 \ldots \eta_n \ldots \phi_1, \phi_2 \ldots \phi_n$ (the number of sets being $(r' - \theta)$,) such that

$$\begin{aligned} &\eta_1x_1 \ldots + \eta_nx_n = \lambda'_{\theta+1}\mathfrak{A}' + \mu'_{\theta+1}\mathfrak{B}' + \ldots \sigma_{\theta+1}\mathfrak{G}' \ldots (17). \\ &\phi_1x_1 \ldots + \phi_nx_n = \lambda'_{r'}\mathfrak{A}' + \mu'_{r'}\mathfrak{B}' + \ldots \sigma'_{r'}\mathfrak{G}'. \end{aligned}$$

From the equations (15) we deduce the $(n-r)$ equations

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \ldots d_{x_n}U \\ a_1, & a_2 \ldots a_n \\ \vdots & \\ \phi_1, & \phi_2 \ldots \phi_n \end{matrix} \right\| = 0 \ldots\ldots\ldots (18).$$

Hence, writing

$$\begin{aligned} &\alpha = \lambda'_1A + \mu'_1B + \ldots \sigma'_1G \ldots\ldots\ldots (19), \\ &\beta = \lambda'_2A + \mu'_2B + \ldots \sigma'_2G, \\ &\vdots \\ &\phi = \lambda'_rA + \mu'_rB + \ldots \sigma'_rG. \end{aligned}$$

And reducing, by the formula (8) of Chap. 1, we have

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \ldots d_{x_n}U \\ A'_1, & A'_2 \ldots A'_n \\ \vdots & \\ G'_1, & G'_2 \ldots G'_n \end{matrix} \right\| = 0 \ldots\ldots\ldots (20).$$

And similarly may the remaining formulæ of (13) be deduced

from the equation (15). Hence the required theorem is demonstrated, a theorem which may be more clearly stated as follows :—

The reciprocals of several systems of equations form together the reciprocal of the equation which is satisfied by the values of the variables which satisfy each of the original systems of equations. (The theorem requires that the number of all the reciprocal equations shall be less than the number of variables.)

Conversely, consider several systems of equations, the whole number of the equations being less than the number of variables. These systems, taken conjointly, have for their reciprocal, the equation which is satisfied by the values satisfying the reciprocal system of each of the given systems.

CHAP. 4. *On some properties of functions of the second order.*

Suppose, as before, $U$ denotes the general function of the second order, or

$$2U = \Sigma(a^2) \cdot x_a^2 + 2\Sigma(a\beta)\, x_a x_\beta \ldots\ldots (21).$$

Also let $V$ denote a function of the second order of the form

$$V = H\left(\left\| \begin{matrix} x_1, x_2 \ldots x_n \\ a_1, a_2 \ldots a_n \\ \vdots \\ x_1, x_2 \ldots x_n \end{matrix} \right\|\right) \quad \ldots\ldots\ldots (22),$$

($H$ being the symbol of a homogeneous function of the second order, and the number $r$ of the quantities $a, \beta \ldots x$, being less than $(n-1)$.) Then $2U - 2kV$, $k$ arbitrary, is of the form

$$\Sigma[a^2] x_a^2 + 2\Sigma[a\beta] x_a x_\beta \ldots\ldots\ldots (23).$$

Suppose $X_1, X_2 \ldots X_n$ determined by the equations

$$[1^2] X_1 + [12] X_2 \ldots + [1n] X_n = 0 \ldots\ldots (24),$$
$$[21] X_1 + [2^2] X_2 \ldots + [2n] X_n = 0,$$
$$\vdots$$
$$[n1] X_1 + [n2] X_2 \ldots + [n^2] X_n = 0 ;$$

equations that involve the condition that $k$ satisfies a certain of the order $(n-r)$, as will be presently proved.

Then shall $x_1 = X_1 \ldots x_n = X_n$ satisfy the system of equations, which is the reciprocal of

$$\left\| \begin{matrix} x_1, x_2 \ldots x_n \\ a_1, a_2 \ldots a_n \\ \vdots \\ x_1, x_2 \ldots x_n \end{matrix} \right\| = 0 \ldots\ldots\ldots\ldots (25).$$

To prove these properties, in the first place, we must find the form of $V$. Consider the quantities $\xi_A$, $\xi_B \ldots \xi^L$, $(n-r)$ in number, of the form

$$\begin{aligned} \xi_A &= A_1x_1 + A_2x_2 \ldots + A_nx_n \ldots\ldots\ldots (26), \\ \xi_B &= B_1x_1 + B_2x_2 \ldots + B_nx_n, \\ &\vdots \\ \xi_L &= L_1x_1 + L_2x_2 \ldots + L_nx_n; \end{aligned}$$

where, if $\Theta$ represent any of the quantities $A, B \ldots L$,

$$\begin{aligned} a_1\Theta_1 + a_2\Theta_2 \ldots + a_n\Theta_n &= 0 \ldots\ldots\ldots (27), \\ \beta_1\Theta_1 + \beta_2\Theta_2 \ldots + \beta_n\Theta_n &= 0, \\ &\vdots \\ x_1\Theta_1 + x_2\Theta_2 \ldots + x_n\Theta_n &= 0. \end{aligned}$$

$$\begin{aligned} 2V &= (A^2)\,\xi_A^2 + (B^2)\,\xi_B^2 + \ldots \\ &\qquad + 2(AB)\,\xi_A\xi_B + \ldots \\ &= \Sigma(A^2)\,\xi_A^2 + 2\Sigma(AB)\,\xi_A\xi_B. \end{aligned}$$

Hence, if $2V = \Sigma\{a^2\}\,x_a^2 + 2\Sigma\{a\beta\}\,\xi_A\xi_B \ldots\ldots\ldots (28)$,

$$\begin{aligned} \{1^2\} &= \Sigma(A^2)\,A_1^2 + 2\Sigma(AB)\,A_1B_1, \\ &\vdots \\ \{12\} &= \Sigma(A^2)\,A_1A_2 + \Sigma(AB)\,(A_1B_2 + A_2B_1), \\ &\vdots \\ [1^2] &= (1^2) - k\{1^2\}, \\ &\vdots \\ [12] &= (12) - k\{12\}. \\ &\vdots \end{aligned}$$

Hence, $\theta$ representing any of the quantities $a, \beta \ldots x$,

$$\begin{aligned} \theta_1\{1^2\} + \theta_2\{12\} \ldots + \theta_n\{1n\} &= 0 \ldots\ldots (29), \\ &\vdots \\ \theta_1\{n1\} + \theta_2\{n2\} \ldots + \theta_n\{n^2\} &= 0; \end{aligned}$$

whence also

$$\begin{aligned} \theta_1[1^2] + \ldots \theta_n[1n] &= \theta_1(1^2) + \ldots \theta_n(1n), \\ &\vdots \\ \theta_n[n1] + \ldots \theta_n[n^2] &= \theta_1(n1) + \ldots \theta_n(n^2). \end{aligned}$$

Hence, the equations for determining $X_1 \ldots X_n$ may be reduced to

$$\begin{aligned} &X_1[a_1(1^2) + \ldots a_n(1n)] + X_2[a_1(21) \ldots + a_n(2n)] \ldots + X_n[a_1(n1) \ldots + a_n(n^2)] = 0 \ldots (30), \\ &X_1[\beta_1(1^2) + \ldots \beta_n(1n)] + X_2[\beta_1(21) \ldots + \beta_n(2n)] \ldots + X_n[\beta_1(n1) \ldots + \beta_n(n^2)] = 0, \\ &\vdots \\ &X_1[n_1(1^2) + \ldots n_n(1n)] + X_2[n_1(21) \ldots + n_n(2n)] \ldots + X_n[n_n(n1) \ldots + n_n(n^2)] = 0. \\ &X_1[r+1, 1] + X_2[r+1, 2] \ldots + X_n[r+1, n] = 0, \\ &\vdots \\ &X_1[n, 1] + X_2[n, 2] \ldots + X_n[n^2] = 0. \end{aligned}$$

Eliminating $X_1 \ldots X_n$, since the first $(r)$ equations do not contain $k$, the equation in this quantity is of the order $(n - r)$.

Next form the reciprocals of the equations (25). These are

$$\left\| \begin{matrix} d_{x_1} U, & d_{x_2} U \ldots d_{x_n} U \\ A_1, & A_2 \ldots A_n \\ \vdots & \\ L_1, & L_2 \ldots L_n \end{matrix} \right\| = 0 \quad \ldots\ldots\ldots \quad (31).$$

From which we may deduce

$$\left\| \begin{matrix} a_1 d_{x_1} U \ldots + a_n d_{x_n} U, & \beta_1 d_{x_1} U \ldots + \beta_n d_x U \ldots x_1 d_{x_1} U \ldots x_n d_{x_n} U, & d_{x_{r+1}} U \ldots d_{xn} U \\ 0 \quad , & 0 \quad \ldots \quad 0 & A_{r+1} \ldots A_n \\ \vdots & & \\ 0 \quad , & 0 \quad \ldots \quad 0 & L_{r+1} \ldots L_n \end{matrix} \right\| = 0 \ldots (32).$$

which is evidently satisfied by $x_1 = X_1$, $x_2 = X_2 \ldots x_n = X_n$.

In the case of four variables, the above investigation demonstrates the following properties of surfaces of the second order.

I. If a cone intersect a surface of the second order, three different cones may be drawn through the curve of intersection, and the vertices of these lie in the plane which is the polar reciprocal of the vertex of the intersecting cone.

II. If two planes intersect a surface of the second order through the curve of intersection, two cones may be drawn, and the vertices of these lie in the line which is the polar reciprocal of the line of intersection of the two planes.

Both these theorems are undoubtedly known, though I am not able to refer for them to any given place.

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## VIII.—ON A QUESTION IN THE THEORY OF PROBABILITIES.

By R. L. Ellis, M.A. Fellow of Trinity College.

The following question affords a good illustration of the methods employed in the more difficult parts of the theory of probabilities. In a paper presented to the Philosophical Society, I applied the kind of analysis we are about to make use of, to the celebrated *Rule of Least Squares*. There is, in fact, a close analogy between the two investigations. Laplace's solution of the present question is obtained by a process similar to that which he had employed when treating of the best method of combining discordant observations.

What is the probability that the sum of the times which each of $n$ persons has respectively yet to live will amount to a given time $T$?

Let $\phi_p x_p dx_p$ be the probability that the $p^{\text{th}}$ person will live precisely a time $x_p$ longer, $\phi_p$ denoting some function of $x_p$, which is necessarily such that

$$\int_0^\infty \phi_p x_p dx_p = 1,$$

as it is certain that he will die at some time or other.

Let $x_1, x_2 \ldots x_n$ be so related that

$$x_1 + x_2 + \ldots x_n = T.$$

The probability of this particular combination is

$$\phi_1 x_1 \phi_2 x_2 \ldots \phi_n x_n dx_1 dx_2 \ldots dx_n,$$

or $$\phi_1 x_1 \phi_2 x_2 \ldots \phi_n (T - x_1 - \ldots x_{n-1})\, dx_1 \ldots dx_n,$$

and the aggregate probability sought is the integral of this expression obtained by giving all possible positive values to $x_1 \ldots x_{n-1}$, which do not make $T - x_1 - \ldots x_{n-1}$ negative. Thus we have

$$P = dx_n \int_0 \ldots \int_0 \phi_1 x_1 \phi_2 x_2 \ldots \phi_n (T - x_1 - \ldots x_{n-1})\, dx_1 \ldots dx_{n-1}.$$

Now, by Fourier's theorem,

$$\phi_n (T - x_1 - \ldots x_{n-1}) = \frac{1}{\pi} \int_0^\infty da \int_0^\infty \phi_n x_n \cos a (T - x_1 - \ldots x_{n-1} - x_n)\, dx_n,$$

$T - x_1 \ldots x_{n-1}$ being supposed to lie between 0 and $\infty$; for all negative values of this quantity, the second member of the equation is equal to zero. Consequently, all the integrations may now be taken from zero to infinity; and thus, since as $T$ and $x_n$ vary together, $dT = dx_n$

$$P = \frac{dT}{\pi} \int_0^\infty da \int_0^\infty dx_1 \ldots \int_0^\infty dx_n \phi_1 x_1 \ldots \phi_n x_n \cos a\, (T - \Sigma x).$$

Now it may be shewn that the greatest value of

$$\int_0^\infty dx_1 \ldots \int_0^\infty dx_n \phi_1 x_1 \ldots \phi_n x_n \cos a\, (T - \Sigma x) \ldots (a)$$

corresponds to $a = 0$, and is, therefore, unity; and that when $n$ is large $(a)$ diminishes rapidly as $a$ increases. Consequently the value of $\int_0^\infty (a)\, da$ depends, when $n$ is very large, on the elements for which $a$ is very small. This consideration enables us to employ an approximate value of $(a)$.

Let $T = t + m$, $m$ being a disposable quantity; then

$$(a) = \cos at \int_0^\infty dx_1 \ldots \int_0^\infty dx_n \phi_1 x_1 \ldots \phi_n x_n \cos a\,(m - \Sigma x)$$
$$+ \sin at \int_0^\infty dx_1 \ldots \int_0^\infty dx_n \phi_1 x_1 \ldots \phi_n x_n \sin a\,(m - \Sigma x),$$

which may be thus written

$$(a) = \cos atG + \sin atH.$$

In order to obtain approximate values of $G$ and $H$, expand $\cos a\,(m - \Sigma x)$ and $\sin a\,(m - \Sigma x)$; they become respectively, $a$ being very small,

$$1 - \tfrac{1}{2} a^2 (m - \Sigma x)^2 \quad \text{and} \quad a\,(m - \Sigma x).$$

Now let $\int_0^\infty \phi x\, x dx = K \quad \int_0^\infty \phi x\, x^2 dx = k^2$;

then, since $\int_0^\infty \phi x dx = 1$, we shall have

$$G = 1 - \tfrac{1}{2} a^2 (m^2 - 2m\Sigma K + 2\Sigma K_1 K_2 + \Sigma k^2),$$
$$H = a\,(m - \Sigma K) \quad \text{approximately.}$$

Let $m = \Sigma K$, then

$$m^2 - 2m\Sigma K = -\{\Sigma K\}^2 = -\Sigma K^2 - 2\Sigma K_1 K_2,$$

and thus

$$G = 1 - \tfrac{1}{2} a^2 \Sigma (k^2 - K^2),$$
$$H = 0;$$

and therefore, while $a$ is very small,

$$(a) = \cos at \left\{1 - \tfrac{1}{2} a^2 \Sigma (k^2 - K^2)\right\}.$$

We have next to show that $\Sigma (k^2 - K^2)$ is a positive quantity.

Consider the definite integral

$$\int_0^\infty \int_0^\infty \phi x \phi z\, (z - x)^2\, dx dz;$$

it is necessarily positive, since every element is so, $\phi x dx$ being the expression of a probability, and therefore essentially positive.

Expanding $(z - x)^2$, we find for the value of this integral

$$k^2 - 2K^2 + k^2 \quad \text{or} \quad 2\,(k^2 - K^2).$$

Hence $k^2 - K^2$, and therefore $\Sigma (k^2 - K^2)$, is positive.

(This demonstration is due to Poisson, *Con. des Tems*, 1827).

Returning to the value we have found for $(a)$, we see that we may in all cases represent $(a)$ by $\cos at\, e^{-\frac{1}{2} a^2 \Sigma (k^2 - K^2)}$, since when $a$ is very small, the two expressions tend to coincide;

and when $a$ is not so, both are sensibly zero, $\Sigma(k^2 - K^2)$ being a large quantity of the order $n$. Consequently

$$\int_0^\infty (a)\, da = \sqrt{\left(\frac{\pi}{2}\right)} \frac{1}{\{\Sigma(k^2 - K^2)\}^{\frac{1}{2}}}\, e^{-\frac{t^2}{2\Sigma(k^2-K^2)}}$$

$$\text{and} \quad P = \frac{dt}{\sqrt{(2\pi)}} \frac{1}{\{\Sigma(k^2 - K^2)\}^{\frac{1}{2}}}\, e^{-\frac{t^2}{2\Sigma(k^2-K^2)}} \ldots\ldots(p),$$

$P$ being the probability that the required sum $T$ shall be precisely equal to $\Sigma K + t$. The greatest value of $P$ corresponds to $t = 0$; consequently the most probable value of $T$ is $\Sigma K$.

It is to be remarked that the approximate formula $(p)$ is independent of the law of probability expressed by the function $\phi x$: it depends merely on the two definite integrals

$$\int_0^\infty x\phi x dx \quad \text{and} \quad \int_0^\infty x^2 \phi x dx.$$

We have stopped the approximation to the value of $G$ at the second power of $a$. Had we gone farther, and retained only the *principal* term in the coefficient of each power of $a$, a similar result, viz. one which may be assumed as coincident with the exponential function, would, there is little doubt, have been obtained; while the coefficient of each power of $a$ in $H$ would be negligible in comparison of the corresponding power in $G$. Some remarks on this, or at least on a cognate question, will be found in the paper already mentioned.

As a verification of the approximation we have employed, which is in effect the same as that of Laplace, let us suppose that the functions $\phi_1$, $\phi_2$...are all of the same form $\phi$, and that $\phi x = e^{-x}$. Then, as we have seen, the required probability is obtained by integrating

$$e^{-x_1 - x_2 \ldots - (T - x_1 - x_2 - \ldots)}\, dx_1 \ldots dx_{n-1},$$

for all positive values of $x_1$, $x_2 \ldots x_{n-1}$ which do not transgress the limits

$$x_1 + x_2 \ldots x_{n-1} = T.$$

Now $$e^{-x_1 - x_2 \ldots - (T - x_1 - x_2 - \ldots)} = e^{-T},$$

and thus we have $P = e^{-T} dT \int_0 dx_1 \ldots \int_0 dx_{n-1}$,

the limits being given by

$$x_1 + x_2 \ldots x_{n-1} \leqq T.$$

Hence, it is easily seen that

$$P = \frac{e^{-T}}{\Gamma(n)}\, T^{n-1} dT.$$

In order to compare this with the approximate expression ($p$), I remark that $T = n - 1$ renders $P$ a maximum; assume, therefore, $T = n - 1 + t$,

$$\text{then } P = \frac{e^{-(n-1)}(n-1)^{n-1}}{\Gamma(n)}\, e^{-t}\left(1 + \frac{t}{n-1}\right)^{n-1} at.$$

Now, by Stirling's theorem, or by that which Binet proposes to substitute for it, (vide *Journal de l'Ecole Polytechnique*, XVI. p. 226), we have, when $n$ is very large,

$$\Gamma(n) - = \sqrt{(2\pi)}\; e^{-(n-1)}(n-1)^{n-\frac{1}{2}}.$$

Consequently

$$\frac{e^{-(n-1)}(n-1)^{n-1}}{\Gamma(n)} = \frac{1}{\sqrt{\{2\pi(n-1)\}}} = \frac{1}{\sqrt{(2\pi n)}} \cdot \ldots \cdot q.p.$$

$$\text{Again } \left\{1 + \frac{t}{n-1}\right\}^{n-1} = 1 + t + 1.\left(1 - \frac{1}{n-1}\right)\frac{t^2}{1.2} + \left(1 - \frac{1}{n-1}\right)\left(1 - \frac{2}{n-1}\right)\frac{t^3}{1.2.3} + \&c.$$

$$= e^t + ft,$$

$ft$ being a certain function of $t$ and $n$. Therefore

$$e^{-t}\left(1 + \frac{t}{n-1}\right)^{n-1} = 1 + e^{-t}ft.$$

Now the coefficient of $t^2$ in $ft$ is

$$-\frac{1}{(n-1).1.2},$$

$$\text{that of } t^3 \text{ is } \left\{-\frac{3}{n-1} + \frac{2}{(n-1)^2}\right\}\frac{1}{1.2.3},$$

and that of $t^4$ is

$$\left\{-\frac{6}{n-1} + \frac{11}{(n-1)^2} - \frac{6}{(n-1)^3}\right\}\frac{1}{1.2.3.4}.$$

Hence the coefficient of $t^2$ in $e^{-t}ft$ is

$$-\frac{1}{2(n-1)},$$

that of $t^3$ is

$$\frac{1}{2(n-1)} - \frac{1}{2(n-1)} + \frac{1}{3(n-1)^2} \text{ or } \frac{1}{3(n-1)^2};$$

and, lastly, that of $t^4$ is

$$-\frac{1}{4(n-1)}+\frac{2}{4(n-1)}-\frac{2}{1.2.3\,(n-1)^2}-\frac{1}{4\,(n-1)}$$
$$+\frac{11}{1.2.3.4}\,\frac{1}{(n-1)^2}-\frac{1}{4\,(n-1)^3},$$

$$\text{or}\quad \frac{1}{1.2}\left\{\frac{1}{2\,(n-1)}\right\}^2-\frac{1}{4\,(n-1)^3}.$$

Now if, in forming the approximate expression, we reject all terms of the form $\left\{\frac{t}{\sqrt{(n)}}\right\}^p \frac{1}{n^q}$, where $q$ is different from zero, *i.e.* if we look on $\frac{t}{\sqrt{(n)}}$ as a quantity all whose powers are to be retained, except when divided by any power of $n$, the value of $e^{-t}\left(1+\frac{t}{n-1}\right)^{n-1}$ may be taken as equal to

$$1-\frac{t^2}{2\,(n-1)}+\frac{1}{1.2}\cdot\frac{t^4}{4\,(n-1)^2}-\&c.$$

which, as similar results would have been obtained had we pursued the investigation farther, might be shown to be equal to

$$e^{-\frac{t^2}{2(n-1)}},$$

and thus

$$P=\frac{1}{\sqrt{(2\pi n)}}\,e^{-\frac{t^2}{2(n-1)}}\,dt,$$

$P$ is the probability that $T$ is equal to $n-1+t$: writing $t+1$ for $t$ in $\frac{t^2}{n-1}$ and reducing, we find that, within the limits of the approximation, it may also be assumed as the probability that $t$ is equal to $n+t$: also $\frac{t^2}{n-1}=\frac{t^2}{n}$ . . . . *q.p.*, and thus

$$P=\frac{1}{\sqrt{(2\pi n)}}\,e^{-\frac{t^2}{2n}}\,dt;$$

which, as in our case, $k^2=2$ and $K=1$ is precisely equivalent to the result deduced from the general formula $(p)$.

The legitimacy of some parts of the preceding approximation may be questioned; as quantities which are neglected may, under certain conditions, be larger than those which are retained: and, as the result coincides with that of the general method, the doubt thus suggested appears to extend to the latter. The subject of approximation by means of definite integrals is certainly not free from obscurity.

The method of this paper extends *m.m.* to the case in which we seek to determine the degree of improbability that the average length of the reigns of a series of kings shall exceed by a given quantity the average deduced from authentic history. The application of considerations of this nature to historical criticism appears to have been first made in Sir Isaac Newton's Chronology. They are doubtless entitled to much attention; but any attempt to evaluate their legitimate influence, would, for more than one reason, be unsatisfactory.

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## IX.—ON THE BALANCE OF THE CHRONOMETER.

It is well known that a common watch goes more slowly when its temperature is raised, and *versâ vice*. The reason of this is that the elasticity of the balance-spring decreases with the increment of temperature and increases with its decrement. Neglecting the mass of the spring and the connection of the balance with the other parts of the watch, we may take as the equation for determining the oscillations of the balance,

$$\frac{d^2\theta}{dt^2} + \frac{e\theta}{I} = 0,$$

where $e$ depends on the form and elasticity of the spring, and $I$ is the moment of inertia of the balance. The time of oscillation depends, of course, on the ratio $\frac{e}{I}$, $e$ being, as we have said, a function of $t$ the temperature. In order, therefore, to the equable rate of the watch, it would be necessary that $I$ should be such a function of $t$, that $\frac{e}{I}$ may be constant. In the balance of a common watch $I$ is sensibly constant. Hence the inequality of which we have spoken.

In the chronometer the balance is so constructed that its figure alters when the temperature varies. The figure (4) represents a common form of the chronometer balance. The arc $AB$, which carries a weight at $C$, is formed of two concentric laminæ of different kinds of metal, the outer lamina being the most expansible. These two laminæ are securely united in their whole length, so that an increase of temperature necessarily distorts the arc $AB$ into some form like $AB'$. Similarly for $ab$. Contrary effects are produced by a decrease of temperature. Thus, the moment of inertia $I$ decreases as $t$ increases; as $e$ also does. And thus we are enabled, by

suitable adjustments, to make $\frac{e}{I}$, at least approximately, constant.

It would, I believe, be impossible, without some hypothesis, to determine the form which $AB$ assumes under the influence of a change of temperature. The following suppositions are probably sufficiently near the truth to be applicable when the variations of $t$ are not excessive.

Let us suppose the laminæ to be cylindrical and concentric, and bounded by four plane surfaces, two of which are perpendicular to the axis of the cylinder, while the other two, which form the boundaries at $A$ and $B$, pass through the axis. These conditions being fulfilled, whatever the value of $t$ may be, it is clear that the variation of form can depend on two elements only, namely the radius of the cylinder, and the angle which $AB$ subtends at its centre. To determine these, we assume that the middle filament of each lamina expands as it would do if free.

In the normal state, let $2\epsilon$, $2\epsilon'$ be the thicknesses of the outer and inner laminæ respectively, $r$ the radius of the boundary of the two laminæ, $\mu$, $\mu'$ the coefficients of expansibility of the outer and inner laminæ ($\mu > \mu'$), $\theta$ the angle subtended at the centre.

The radii of the middle filaments are, therefore, $r+\epsilon$, $r-\epsilon'$; let their lengths be $l$ and $l'$, then

$$l = (r + \epsilon)\,\theta, \quad l' = (r - \epsilon')\,\theta.$$

For an increase of temperature $t$, let $r$ and $\theta$ become $r_1$ and $\theta_1$: then we shall have

$$l\,(1 + \mu t) = (r_1 + \epsilon)\,\theta_1, \quad l'\,(1 + \mu' t) = (r_1 - \epsilon')\,\theta_1;$$

$\epsilon$ and $\epsilon'$ being so small that their variations may be neglected.

Hence
$$\frac{r_1 + \epsilon}{r_1 - \epsilon'} = \frac{l}{l'}\,\frac{1 + \mu t}{1 + \mu' t},$$

and therefore
$$r_1 - \epsilon' = (\epsilon + \epsilon')\,\frac{l'\,(1 + \mu t)}{l\,(1 + \mu t) - l'\,(1 + \mu' t)},$$

which, as $\mu$ and $\mu'$ are very small, is approximately

$$r_1 - \epsilon' = (\epsilon + \epsilon')\,\frac{l'}{l - l'}\left\{1 - (\mu - \mu')\,\frac{l'}{l - l'}\,t\right\}.$$

When $t = 0$
$$r - \epsilon' = (\epsilon + \epsilon')\,\frac{l'}{l - l'}.$$

Consequently, for a first approximation,

$$\Delta r = -\,\Delta\mu\,\frac{r^2}{\tau}\,t \ldots\ldots\ldots\ldots\ldots (1),$$

where $\Delta r = r_1 - r$, $\Delta\mu = \mu - \mu'$, and $\tau = \epsilon + \epsilon_1$.

Again $\tau\theta_1 = l - l' + (\mu l - \mu' l')\, t.$

When $t = 0$ $\tau\theta = l - l';$

and therefore $\tau\Delta\theta = (\mu l - \mu' l')\, t.$

But $\mu l - \mu' l' = r\theta\Delta\mu + (\mu\varepsilon - \mu'\varepsilon')\,\theta,$

and the last term is negligible. Therefore

$$\Delta\theta = \frac{r}{\tau}\,\theta\Delta\mu t \ldots\ldots\ldots\ldots\ldots (2).$$

We distinctly perceive from (1) and (2) why the effects of *distortion* are so considerable in comparison with those of simple expansion; it is because the expressions of $\Delta r$ and $\Delta\theta$ have the small quantity $\tau$ in the denominator. $AB$ becomes a larger arc of a smaller circle.

To apply these results: we suppose that when $t = O$ the centre of $AB$ coincides with the central point $O$; and $AB$, being securely fastened at $A$, continues perpendicular at that point to the line $OA$, consequently its centre remains in that line. Let $O'$ be its new position, then $OO' = -\Delta r$. If $m_1$ be the mass of $AB$, its moment of inertia about $O$ was $m_1 r^2$; about $O'$ it is $m_1 r^2 - 2m_1\Delta\mu\,\frac{r^3}{\tau}\,t$ nearly. Let $G$ be the centre of gravity of $AB'$; then, in the triangle $OO'G$, we have $OG^2 = OO'^2 + (O'G)^2 + 2OO'.O'G\cos\frac{1}{2}\theta_1$, since $\angle GO'A = \frac{1}{2}\theta$. $OO'^2$ or $(\Delta r)^2$ may be neglected, then, approximately,

$$(OG)^2 = (O'G)^2 - 2\Delta r.O'G\cos\tfrac{1}{2}\theta,$$

and, as $$O'G = r_1\,\frac{\sin\frac{1}{2}\theta_1}{\frac{1}{2}\theta_1} = 2r\,\frac{\sin\frac{1}{2}\theta}{\theta}$$ nearly,

we have $$(OG)^2 - (O'G)^2 = -\,2r\,\frac{\sin\theta}{\theta}\,\Delta r.$$

Now the moment of inertia round $O$ is equal to that round $O'$ increased by $m\{(OG)^2 - (O'G)^2\}$; hence, finally,

$$\Delta I_1 = -\,2m_1\Delta\mu\,\frac{r^3}{\tau}\,t\left(1 - \frac{\sin\theta}{\theta}\right)\ldots\ldots\ldots(3),$$

$I_1$ being the moment of inertia of the arc $AB$.

(In accordance with the rest of the approximation the expansion of $AO$ is neglected).

Again, we will suppose the weight at $C$ to be a material particle, and that the angle $AOC$ is equal to $\phi$. Then, $I_2$ being the moment of inertia of this weight, whose mass we will denote by $m_2$, we shall have

$$\Delta I_2 = -\,2m_2\Delta\mu\,\frac{r^3}{\tau}\,t\,(1 - \cos\phi)\ldots\ldots\ldots(4),$$

Consequently, as the inertia of the bar $OA$ does not undergo any sensible alteration, and as every thing which has been proved of $OAB$ is true of $Oab$, we have, finally,

$$\Delta I = -4\Delta\mu \frac{r^3}{\tau} t\left\{m_1\left(1 - \frac{\sin\theta}{\theta}\right) + m_2(1 - \cos\phi)\right\} \ldots (5).$$

It appears that the variation of $e$ is exactly proportional to $t$: so that $e$ becomes $e(1 - \nu t)$, $\nu$ being some constant. Consequently we must have, in order that $\frac{e(1-\nu t)}{I + \Delta I} = \frac{e}{I}$,

$$\nu I = 4\Delta\mu \frac{r^3}{\tau}\left\{m_1\left(1 - \frac{\sin\theta}{\theta}\right) + m_2(1 - \cos\phi)\right\}. \ldots . (6).$$

In calculating the value of $I$ we may take into account the moment of inertia of $aOA$; moreover, instead of the approximate expression $m_1 r^2$ for the moment of inertia of $AB$, we may employ a more accurate one involving the quantity $\varepsilon$ and $\varepsilon'$; the approximate expression is sufficiently accurate for the determination of $\Delta I$.

The adjustment for compensation is effected by shifting the weight $m_2$ along $AB$; that is, by altering the value of $\phi$ until (6) is fulfilled. On the hypothesis we have made, the value of $I$ for $t = 0$ is not affected by the change of $\phi$.

In determining the approximate expressions (5) and (6), we have neglected all terms in which $\Delta\mu$ occurs not divided by $\tau$; all terms involving $\Delta\mu$ multiplied by $\varepsilon$ or $\varepsilon'$; all terms into which any power of $\mu$ or $\mu'$ enters. In consequence of the last restriction $t$ can only rise to the first power in the result. If this were absolutely correct it would follow that, if the compensation were effected for a particular value of $t$, it would subsist accurately for all values of $t$. For instance, if we give $t$ equal values, positive and negative, the decrease of $I$ in the one case ought to be equal to its increase in the other. But when $t$ is considerable, it is found that there is a sensible deviation from this result; and, assuming that the expression for $\Delta e$ does not in any perceptible manner involve powers of $t$, it follows that that of $\Delta I$ must do so. Any term involving $t^2$ (and, *a fortiori*, any higher powers of that quantity), must be very small, since $t$ always occurs multiplied by $\mu$ or $\mu'$; but it may, nevertheless, sensibly affect the chronometer's daily rate. On the usual construction, the balance oscillates 216,000 times in twenty-four hours. Consequently a very slight change in the moment of inertia of the balance will become perceptible in that period.

In order to obviate the consequent error, it has been proposed by Mr. Dent, a distinguished chronometer-maker of the present day, to alter the form of the balance. Fig. (5) represents one of those which he proposes to substitute for that in common use. It would be easy to determine the corresponding expression for $\Delta I$, to the degree of approximation of our previous results. As, however, the comparison of the merits of the two forms must depend on the terms involving $t^2$, it may be well to reserve it for another opportunity. If there appears reason to believe that our hypotheses represent the facts with sufficient accuracy to encourage us to proceed farther, I hope to resume the subject in the next number of the *Journal*. R. L. E.

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## X.—ON A PROBLEM IN PRECESSION AND NUTATION.

In Professor Airy's *Tract on Precession and Nutation*, p. 203, there is an investigation of the angular acceleration which the disturbing force of the Sun tends to impress upon the Earth's mass about an equatoreal diameter (the axis of $z$) at right angles to the plane through the axis of the Earth (the axis of $x$) and the line joining the centre of the Earth with the Sun. The mass is divided into elementary parallelopipeds $dxdydz$, the axis of $y$ being at right angles to those of $x$ and $z$: the integrations are performed in the order of $y$, $x$, $z$. Since, however, the computation may be effected rather more elegantly by taking the integrations in the order of $z$, $y$, $x$, or by conceiving the spheroid to be made up of circular instead of elliptical slices, the method of integration given in this note may not be without interest to students in physical astronomy.

Let $A$ (fig. 6) be the Earth's centre, $AB$ the semi-axis of the spheroid, $S$ the attracting body. Let $P$ be any point of the Earth in the plane $BAC$; draw $PM$ perpendicular to $AB$, and $PN$ to $SA$.

The accelerating force on a particle of the Earth at $P$ is equal to $\frac{S}{SP^2}$ in the direction $PS$, which is equivalent to

$$\frac{S}{SP^2} \cdot \frac{SN}{SP}, \text{ parallel to } AS \ldots\ldots\ldots (1),$$

and to

$$\frac{S}{SP^2} \cdot \frac{PN}{SP}, \text{ parallel to } PN \ldots\ldots\ldots (2).$$

Also the accelerating force on $A$, the centre of the Earth, is equal to

$$\frac{S}{SA^2}, \text{ in } AS \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Let $SA = r$; then, considering the centre of the Earth to be at rest, the disturbing force on $P$ will be, from (1) and (3), approximately,

$$\frac{S}{(r - AN)^2} - \frac{S}{r^2} = \frac{2S.AN}{r^3}, \text{ parallel to } AS;$$

and, from (2), $\dfrac{S.PN}{r^3}$, parallel to $PN$.

Hence the moment of the disturbing force on a particle $m$ at $P$ about a line through $A$, perpendicular to $PAS$, is

$$\frac{3S}{r^3} . m . AN . PN.$$

The same is true approximately for every molecule of the spheroid of which the projection on the plane $BAC$ coincides with $P$.

Let $AC = a$, $AB = c$, $\angle BAS = \theta$, $AM = x$, $MP = y$, $2z$ = length of chord through $P$ at right angles to plane $BAC$. Then $AN = x \cos\theta - y \sin\theta$, $PN = x \sin\theta + y \cos\theta$; and thus we see that the moment of the disturbing force for the whole spheroid is equal to

$$\frac{3S.k}{r^3}\int_{-c}^{+c}\int_{-y'}^{+y'} \{(x^2 - y^2)\sin\theta\cos\theta + xy(\cos^2\theta - \sin^2\theta)\}\, dx . 2z\, dy,$$

where $y'$ = the radius of the circular section through $MP$, and $k$ = the density.

It is evident that the second term of this integral must be zero, because for every $(+ y)$ there is a $(- y)$, $zdy$ being the same for both; hence the required moment

$$= \frac{3S.k}{r^3} . 2 \sin\theta \cos\theta \int_{-c}^{+c}\int_{-y'}^{+y'} (x^2 - y^2)\, z\, dx\, dy.$$

Now $$\int_{-y'}^{+y'} z\, dy = \int_{-y'}^{+y'} (y'^2 - y^2)^{\frac{1}{2}}\, dy = \tfrac{1}{2}\pi y'^2;$$

and, as may easily be ascertained,

$$\int_{-y'}^{+y'} zy^2 dy = \int_{-y'}^{+y'} y^2 (y'^2 - y^2)^{\frac{1}{2}}\, dy = \tfrac{1}{8}\pi y'^4.$$

Hence the double integral is equal to

$$\int_{-c}^{+c} (x^2 . \tfrac{1}{2}\pi y'^2 - \tfrac{1}{8}\pi y'^4)\, dx$$

$$= \tfrac{1}{8}\pi \int_{-c}^{+c} y'^2 (4x^2 - y'^2)\, dx$$

$$= \tfrac{1}{8}\pi \frac{a^2}{c^2} \int_{-c}^{+c} (c^2 - x^2) \left\{ 4x^2 - \frac{a^2}{c^2}(c^2 - x^2) \right\} dx$$

$$= \frac{\pi a^2}{8c^2} \int_{-c}^{+c} \left\{ 4c^2x^2 - 4x^4 - \frac{a^2}{c^2}(c^4 - 2c^2x^2 + x^4) \right\} dx$$

$$= \frac{\pi a^2}{8c^2} \left( \tfrac{8}{3} c^5 - \tfrac{8}{5} c^5 - 2a^2c^3 + \tfrac{4}{3} a^2c^3 - \tfrac{2}{5} a^2c^3 \right)$$

$$= \frac{\pi a^2}{8} c . \frac{16}{15} (c^2 - a^2) = \frac{2\pi}{15} a^2c\,(c^2 - a^2).$$

Hence the required moment

$$= \frac{3S}{r^3} \cdot \frac{4\pi}{15} ka^2c\,(c^2 - a^2) \sin\theta \cos\theta.$$

The sign of this result shews that the effect of the Sun's attraction tends to *increase* the angle $\theta$.

The moment of inertia of the spheroid about the axis through $A$, perpendicular to the area $BAC$, is (we know) equal to

$$\frac{4\pi}{15} ka^2c\,(a^2 + c^2).$$

Hence the angular acceleration is equal to

$$\frac{3S}{r^3} \cdot \frac{a^2 - c^2}{a^2 + c^2} . \sin\theta . \cos\theta.$$

W. W.

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## XI.—NOTES ON MAGNETISM. NO. II.

By R. L. Ellis, M A. Fellow of Trinity College.

In order to a distinct understanding of the results obtained in the last number of the *Journal*, it will be desirable to consider the established conventions with respect to the signs of the symbols which we had occasion to employ.

North magnetism is assumed to be positive; hence, of course, south magnetism must be considered as negative.

The measure $M$ of the magnetic power of a bar magnet is, as we have seen, equal to $\int \mu s\, ds$, $\mu$ being the magnetism of the element $ds$, which is situated at a distance from the origin equal to $s$. The limits of the integral are such as to include the whole length of the magnet.

The position of the origin is arbitrary: we may conveniently place it at the centre of the magnet, but the value of

$\int \mu s ds$ is the same whether this be done or any other point be taken. For let the origin be shifted through a distance $a$, so that $s = s' - a$, then

$$\int \mu s ds = \int \mu (s' - a)\, ds' = \int \mu s' ds' - a \int \mu ds' :$$

and as all the integrals extend throughout the length of the magnet $\int \mu ds' = 0$, and therefore

$$\int \mu s ds = \int \mu s' ds' \quad \text{or} \quad M = M',$$

which was to be proved.

But the value of $\int \mu s ds$ changes its sign if the direction in which $s$ is measured changes. Let $l$ be the length of the magnet; then, $s$ being measured in one direction, say from left to right, we have

$$M = \int_0^l \mu s ds.$$

Now suppose that $s' = l - s$, then the limits are interchanged and $ds' = - ds$; consequently

$$M = \int_0^l \mu (l - s')\, ds' = - \int_0^l \mu s' ds' = - M',$$

$s'$ being measured in the direction opposite to that of $s$, or from right to left.

The magnetism of a magnet may thus be always represented by a positive quantity.

Any two points in the axis of a magnet may be taken as its poles. But although the position of the poles is matter of convention, yet relatively to one another, one is the north and the other the south pole.

The physical character by which they are distinguished is this: if a particle of north magnetism be placed in the prolongation of the axis from south to north, it is repelled from the magnet. Contrariwise, if it be placed in the prolongation of the axis towards the south. Further, we must integrate $\mu s ds$ from south to north, *i.e.* $s$ must be taken as positive when $\mu ds$ lies to the north of the origin, in order that $M$ may be positive. This may be shown by supposing a particle of north magnetism $m$ placed in the prolongation of the axis towards the north, and at a distance $r$ from the centre of the magnet. If we assume that from south to north is positive, the action of the magnet on $m$ is $\frac{2Mm}{r^3}$; and as this action is repulsive its expression will be positive, and therefore $M$ is so. If we had assumed from north to south to be positive, the action of the magnet would have been represented by $- \frac{2Mm}{r^3}$,

and as this is positive, $M$ will necessarily be negative. So that, in order to make the measure of the magnet's power positive, we must take the direction $S...N$ as positive.

Consequently the angle $\theta$ must be measured from it. We suppose it measured in the usual manner, viz. in the *unscrew* direction.

The general expression for the moment of rotation due to the action of one magnet on another is much simplified when the two magnets are supposed to lie in one plane.

The dihedral angle $\chi$ is then zero, and consequently the equation

$$L = \frac{MM'}{R^3}\{1 + 3\cos^2\theta - (2\cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\chi)^2\}^{\frac{1}{2}}$$

becomes

$$L = \frac{MM'}{R^3}\{1 + 3\cos^2\theta - 4\cos^2\theta(1 - \sin^2\theta') + 4\sin\theta'\cos\theta\sin\theta\cos\theta' - \sin^2\theta\sin^2\theta'\}^{\frac{1}{2}}.$$

The quantity between the brackets is equal to

$$4\sin^2\theta'\cos^2\theta + 4\sin\theta'\cos\theta\,.\,\sin\theta\cos\theta' + \sin^2\theta\cos^2\theta'.$$

Consequently

$$L = \frac{MM'}{R^3}(\sin\theta\cos\theta' + 2\sin\theta'\cos\theta).$$

This result may be readily established by an independent process, which the reader will find no difficulty in supplying. The last result may be put in the following form:

$$L = \frac{MM'}{2R^3}\{3\sin(\theta + \theta') - \sin(\theta - \theta')\}.$$

Professor Lloyd, in the 19th volume of the *Memoirs of the Royal Irish Academy*, has investigated this case of the mutual action of two magnets. His result is, (mutatis mutandis)

$$L = \frac{MM'}{2R^3}\{\sin(\theta + \theta') - 3\sin(\theta - \theta')\}.$$

This differs from the last written result, merely because, in the Professor's analysis, $\theta$ and $\theta'$ are measured in opposite directions. If we replace $\theta$ in Prof. Lloyd's result by $2\pi - \theta$, it becomes

$$L = \frac{MM'}{2R^3}\{3\sin(\theta + \theta') - \sin(\theta - \theta')\},$$

as before.

The general formula affords a simple solution of the following problem. The position of a magnet, and that of the centre of a needle being given, to place the needle in the

position in which the moment of rotation due to the action of the magnet is a maximum.

By the formula established in the last number of the *Journal*, we have

$$L = \frac{MM'}{R^3} \{1 + 3\cos^2\theta - (3\cos\theta\cos\theta' - \cos\phi)^2\}^{\frac{1}{2}}.$$

We suppose the magnet and needle to be in the same plane. In fig. (7) let $O$ be the centre, $SON$ the line of the axis of the magnet, $C$ the centre of the needle. Project $C$ on $ON$ in $D$, take $OE = 2OD$, draw $EN'$ perpendicular to $SN$ meeting $ON'$, which is at right angles to $OC$ in $N'$, $CN'$ is the line in which the axis of the needle must be placed, its north pole being turned towards $N'$.

In order to prove this, we have only to remark that the angle $\theta$ or $CON$ is constant, the position of $C$ being given; consequently the condition to be fulfilled, in order that the moment of rotation $L$ may be a maximum, is

$$3\cos\theta\cos\theta' - \cos\phi = 0.$$

Now as $\phi$ is the angle between $N'C$ and $SN$, we have

$$ED = CN'\cos\phi,$$

and therefore $OD = \frac{1}{3} CN'\cos\phi$.

Again, $OC = CN'\cos\theta'$ and $OD = OC\cos\theta$.

Hence $OD = CN'\cos\theta\cos\theta'$.

Consequently $3\cos\theta\cos\theta' - \cos\phi = 0$,

or the required condition is fulfilled.

There are three particular cases worth noticing:

(1) $\theta = 0$. In this case $C$ lies in the axis $ON$, $D$ coincides with it, and $ON'$ is perpendicular to $ON$, and therefore parallel to $EN'$. Consequently the point $N'$ is removed to an infinite distance, and $CN'$ is therefore perpendicular to $ON$. The corresponding value of $L$ is $\frac{2MM'}{R^3}$, and is the maximum maximorum.

(2) $\theta = \theta'$. In this case the magnet and needle are parallel to one another. The quadrilateral $CDEN'$ is a parallelogram, $EN'$ is equal to $DC$, and consequently

$$\tan COD : \tan N'OE :: EO : OD :: 2 : 1.$$

But $COD = \theta$ and $N'OE = \frac{\pi}{2} - \theta$,

since $CON'$ is a right angle. Consequently

$$\tan\theta = 2\cot\theta,$$

or $\tan\theta = \sqrt{2}$.

The corresponding value of $3\cos^2\theta$ is therefore unity; and consequently we have, in this case,

$$L = \sqrt{2}\,\frac{MM'}{R^3} \quad (\theta = 54^\circ\ 44'\ 8'').$$

(3) $\theta = \frac{\pi}{2}$. Here $D$ (and therefore $E$) coincides with $O$, while $ON'$ lies in the axis $OS''$. Consequently $N'$ is at $O$, and the needle is therefore again perpendicular to the magnet. In this case

$$L = \frac{MM'}{R^3},$$

The first and third cases were noticed in the last number of the *Journal*. In the second, the value of $L$ is a mean proportional between what is in the other two cases, in the last of which it is a minimum maximorum.

If we were required, for a given position of $C$, to find the position in which the needle would be in equilibrium, or the moment $L$ equal to zero, we might have recourse to Gauss's construction already mentioned; for if the needle be placed along the line in which the magnet tends to attract or repel $C$, as the dimensions of the needle are small, every element would approximately be attracted or repelled along this line, and therefore the total action would be destroyed by the resistance of $C$.

Thus there are always two directions for every position of $C'$; one of maximum moment and the other of equilibrium: these two directions are at right angles to one another.

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## XII.—MATHEMATICAL NOTE.

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx.$$

This definite integral is evaluated in a curious manner by M. Bertrand in *Liouville's Journal*. The demonstration I am about to give of his result, is somewhat different in form from that which he made use of.

The method employed in an ingenious paper which appeared in the last volume of the *Journal* (III. p. 188), will apply to the integral we are about to consider.

Let $$fu = \int_0^1 \frac{\log(1+ux)}{1+x^2}\,dx.$$

Then $$\frac{d}{du}fu = \int_0^1 \frac{x\,dx}{(1+ux)(1+x^2)}.$$

Now $$\frac{x}{1+x^2}+\frac{u}{1+u^2}=(1+ux)\,\frac{x+u}{(1+x^2)(1+u^2)}.$$

Consequently

$$\frac{x}{(1+ux)(1+x^2)}=\frac{x+u}{(1+x^2)(1+u^2)}-\frac{u}{(1+ux)(1+u^2)},$$

and therefore

$$\frac{d}{du}fu=\frac{1}{1+u^2}\int_0^1\frac{x\,dx}{1+x^2}+\frac{u}{1+u^2}\int_0^1\frac{dx}{1+x^2}-\frac{u}{1+u^2}\int_0^1\frac{dx}{1+ux}$$

$$=\tfrac{1}{2}\log 2\;\frac{1}{1+u^2}+\frac{\pi}{4}\;\frac{u}{1+u^2}-\frac{\log(1+u)}{1+u^2}.$$

Integrate for $u$, from 0 to 1,

$$f(1)-f(0)=\frac{\pi}{8}\log 2+\frac{\pi}{8}\log 2-f(1).$$

But $f(0)=0$, since $\log 1=0$. Therefore

$$f(1)=\frac{\pi}{8}\log 2.$$

But $$f(1)=\int_0^1\frac{\log(1+x)}{1+x^2}\,dx.$$

Therefore $$\int_0^1\frac{\log(1+x)}{1+x^2}\,dx=\frac{\pi}{8}\log 2.$$

The singularity of this method, and its applicability in other cases give it interest: but, as the writer of the paper already noticed pointed out to me, the integral may be got by assuming $x=\tan y$; it then becomes

$$\int_0^{\frac{\pi}{4}}\log(1+\tan y)\,dy,$$

and, by his fundamental equation,

$$\int_0^{\frac{\pi}{4}}\log(1+\tan y)\,dy=\int_0^{\frac{\pi}{4}}\log\left\{1+\tan\left(\frac{\pi}{4}-y\right)\right\}dy$$

$$1+\tan\left(\frac{\pi}{4}-y\right)=1+\frac{1-\tan y}{1+\tan y}=\frac{2}{1+\tan y};$$

and therefore

$$2\int_0^{\frac{\pi}{4}}\log(1+\tan y)\,dy=\frac{\pi}{4}\log 2,$$

whence the truth of M. Bertrand's result is obvious. ε.

Fig. 1.

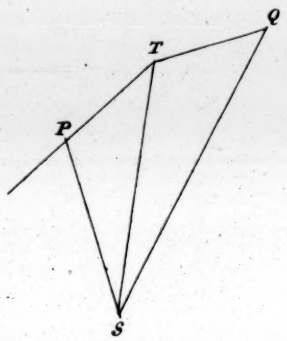



Fig. 2.

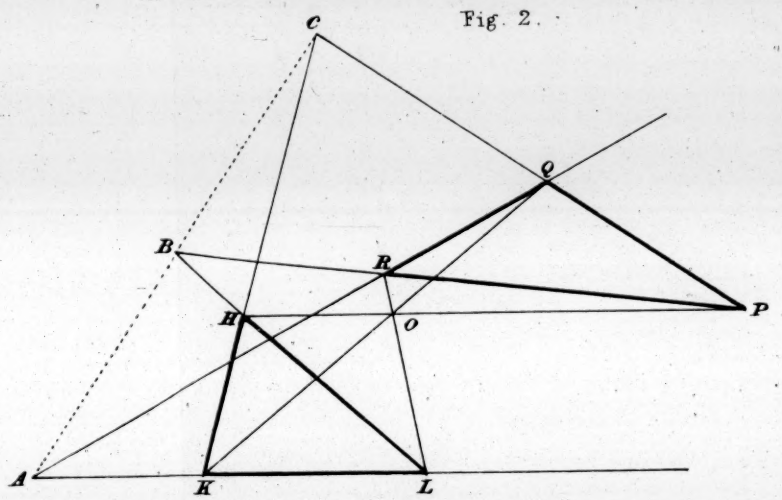



Fig. 3.

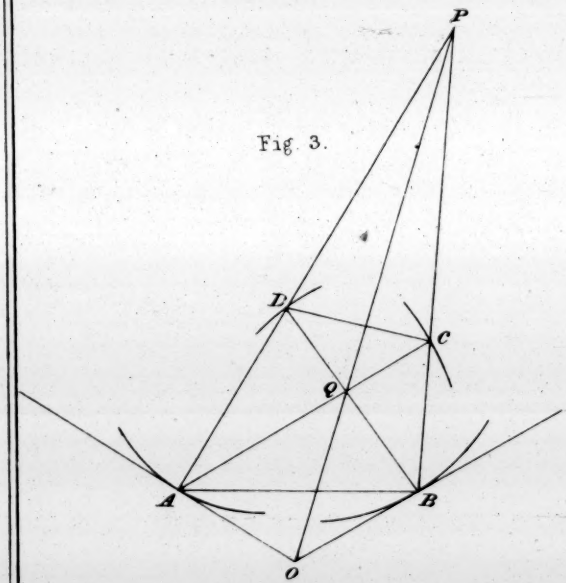



Fig. 4

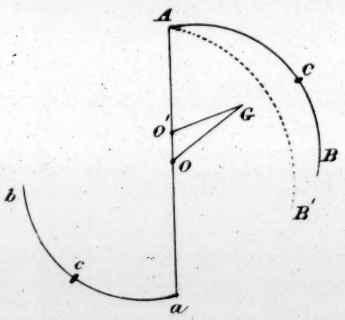



Fig. 5.

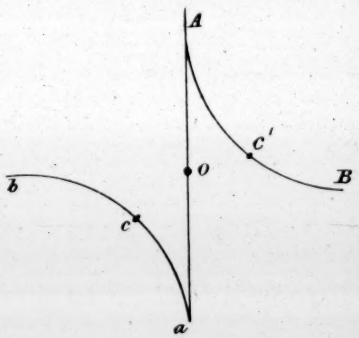



Fig. 6.

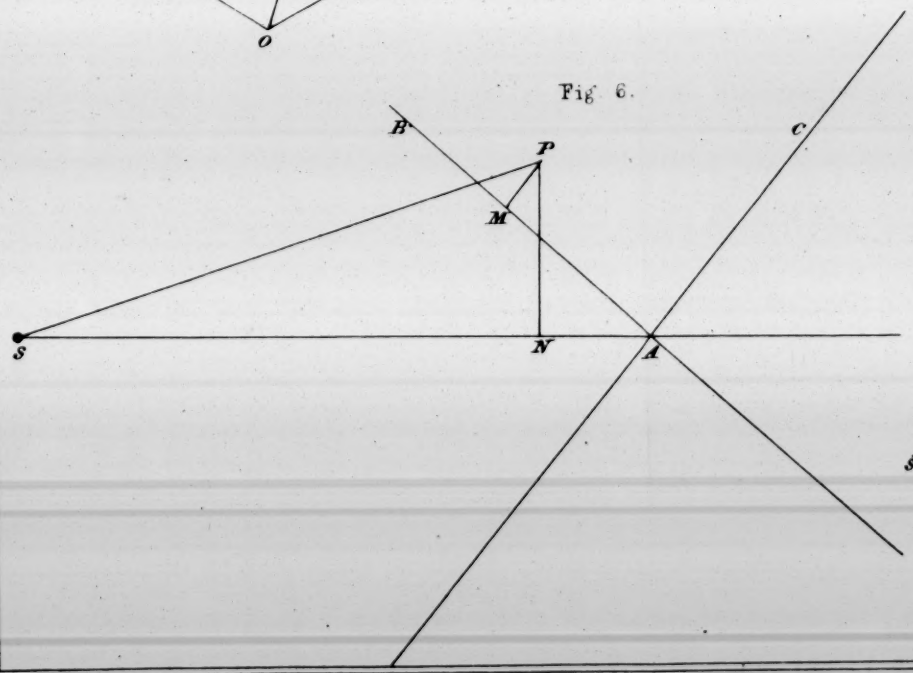
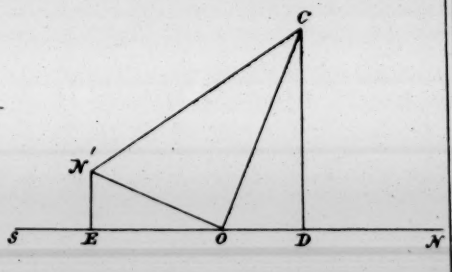



Fig. 7.




Metcalfe & Palmer, Litho